Mathemagical Meandering 01

Bob Albrecht & George Firedrake

⚐⚐⚐Under construction: Version 2012-06-29 ⚐⚐⚐

Ahoy Reader,

This is a work in progress, under construction, ongoing, probably never to be finished. We want to share it with you as we write it, and hope that you will find it useful for learning and teaching, share it with students, teachers, tutors, and others, add stuff that didn't occur to us, fixx our mistrakes, et cetera, et cetera.

Several of the articles in this book were first posted on
the Oregon Council of Teachers of Mathematics listserv.

To download this book and other books (free) as PDF files or Word files or both, go to

http://i-a-e.org/downloads/cat_view/86-free-ebooks-by-bob-albrecht.html

⚐⚐⚐	**We will add new stuff now and then, so check back every month or so.**	⚐⚐⚐

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The Plop Investigation | TOC

We sent this email circa 2004 to the Oregon Teachers of Mathematics (OCTM) listserv in response to a posting about an interesting "real-life" algebra application.

Good Cheer Mathemagicians,

The "What does math have to do with real life?" posting was delicious, especially the **milkshake plop investigation**. The article "N.C. schools to bring math down to earth" quoted a math teacher describing a real-life application of algebra:

> "For example, a milkshake could illustrate slope," she said. "Teachers would explain how plopping scoops of ice cream into milk makes a milkshake grow. Numbers would be assigned to the scoops of ice cream and the increasing volume of the milkshake. Then, an inclining line could be graphed. That way, when students found the slope of that line, they would see that the answer is more than just a number. It is actually the rate at which the milkshake grew."

As writers of intertwingled math and science instructional stuff, we were intrigued by the many plopifications of this investigation.

The teacher said, "Teachers would explain." Alas, alack, and oh heck. She did not say, "Students would investigate, experiment, discover, learn, et cetera, et cetera."

Since this was given as an example of a **real-life** application, we assume that it was observed many times in real-life places and that many real-life experiments were run to validate the experimental method and results.

The Milkshake Investigation: Adventures in Plopland

We assume that the teacher intends to model the volume of the milkshake as a linear function. The **domain** of the function is measured in scoops, which are easy to count.

> Domain = {no scoop, 1 scoop, 2 scoops, 3 scoops, ..., *n* scoops}

The domain is a finite set of whole numbers, unless you measure tenths of a scoop, hundredths of a scoop, and so on.

The **range** is a measure of the volume of the milkshake. How is it measured? In cubic centimeters? In liters? Do all scoops have the same shape? Do all scoops have the same volume? Do all scoops have the same mass? More about that after we examine milkshake-making mechanics.

Milkshake-making mechanics. Is the ice cream plopped in before adding milk, or is the milk put in first and the scoops of ice cream plopped into the milk? Either way, the milk and ice cream are then mixed in a blender.

Ice cream first, then milk. If the ice cream is plopped in first, then the volume V of ice cream in the milkshake container is a direct variation function of the number of scoops.

$V = (V_{\text{scoop}})$(number of scoops), where V_{scoop} is the volume per scoop of ice cream.

If we identify volume with y and number of scoops of ice cream with x, the y-intercept is 0 and the slope is V_{scoop} measured in appropriate units.

If all scoops have exactly the same volume, say V_{scoop}, the range is

Range = $\{0\ V_{\text{scoop}}, 1\ V_{\text{scoop}}, 2\ V_{\text{scoop}}, 3\ V_{\text{scoop}}, ..., n\ V_{\text{scoop}}\}$

Suppose that all scoops are spheres exactly 5 centimeters in diameter. Then the volume is – well, you can work that out and restate the range in cubic centimeters.

Oops, real-life trouble! A scoop of ice cream is probably malleable. The first scoop hits the bottom of the container, compresses a bit, squeezes out a little air, and changes shape, volume, and density.

Now plop in another scoop. The second scoop plops down on the first scoop. Both change shape, volume, and density, so the volume of ice cream in the container might not be the sum of the volumes of the two scoops before they were plopped.

Changes in shape, volume, and density continue as more scoops are plopped. Soft ice cream will deform more than hard ice cream.

What to do? Measure the volume of each scoop before plopping it into the container? Or somehow measure the volume of ice cream in the container? If the ice cream is soft enough to act like a liquid and conform to the shape of the container, we can calibrate the container to give the volume as a function of height. Easy if the container is a cylinder – more difficult for most milkshake containers we have seen that vary in diameter from bottom to top. Use graduated cylinders or beakers as the milkshake containers? Can we sell that to, say, Baskin-Robbins?

Milk first, then ice cream. If the milk is poured in first and then the ice cream is plopped into the milk, the volume in the container can be modeled by the equation

$V = (V_{\text{scoop}})$(number of scoops) + V_{milk}

where V_{scoop} is the volume per scoop of ice cream and V_{milk} is the volume of milk.

If we identify volume with y and number of scoops with x, the y-intercept is V_{milk} and the slope is V_{scoop}, both measured in appropriate units.

Conjecture: Ice cream plopped into milk will not deform as much as ice cream plopped into an empty container. It will be more likely to retain its original shape and volume.

Practical problems: Is the volume of milk measured before it is poured into the container? Or is it measured after it is in the container, perhaps by calibrating the container with a volume versus height scale? Is the density of ice cream greater than the density of milk so that the scoops submerge as they plop into the container? Or is the density of ice cream less that the density of milk so that part of the ice cream floats above the surface of the milk?

How about using mass instead of volume? We suggest that it might be easier to measure mass than measure volume. The mass M of a scoop of ice cream doesn't change when the shape and volume change. Using mass instead of volume, the equations are

Ice cream, then milk:

Plop in ice cream: $M = (M_{\text{scoop}})(\text{number of scoops})$

where M_{scoop} = mass per scoop, M_{milk} = mass of milk

Then add milk: $M = (M_{\text{scoop}})(\text{number of scoops}) + M_{\text{milk}}$

where M_{scoop} = mass of ice cream per scoop, M_{milk} = mass of milk

Milk, then ice cream:

Pour in milk: $M = M_{\text{milk}}$

where M_{milk} = mass of milk

Then add ice cream: $M = (M_{\text{scoop}})(\text{number of scoops}) + M_{\text{milk}}$

where M_{scoop} = mass of ice cream per scoop, M_{milk} = mass of milk

Our Real-Life Milkshake Plop Quest

Where is this "real-life" stuff actually used in real life? "Eureka!" We exclaimed as we ran through the streets of Newport fully clothed (too cold to run naked as did Archimedes) and wended our way to places that make milkshakes.

Alas, no joy. In every place where people make milkshakes, not one place measured the volume or mass of the milkshake as a function of the number of scoops of ice cream plus the milk. Indeed, it was a bit scary as these real-life people backed away from our inquiry and reached for the telephone.

Rebuked by real-life people who strangely don't know about this real-life application of algebra, we hiked many kilometers in the wonderment of Gaia, thinking about all of the above, and sat on a rock with a great view of the ocean. Suddenly there plopped into our mind an incredible eureka about ... but that's another story for another time.

A Plethora of Snakes? | TOC

We sometimes tutor high school students who failed the state math test prior to their retaking the test. As Tom Lehrer once exhorted, we want to be prepared, so we downloaded a sample test and spent some happy time working it.

We found the following problem rather interesting.

There are 6 snakes in a certain valley. The population of snakes doubles every year. In how many years will there be 96 snakes?

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A. 2    B. 3    C. 4    D. 8
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Here's a handy table showing the population growth of the snakes in that certain valley.

Year	Number of snakes
0	6
1	12
2	24
3	48
4	96

The table quickly gives the answer: C. 4 years. The test taker does not need to know that this population growth can be modeled by the equation: snakes = 6×2^n where n = number of years.

Let's keep on truckin':

Number of snakes = 6×2^n	
Year	Number of snakes
5	192
10	6144
20	6,291,456
30	6,442,450,944
40	Your turn. please calculate.

How big is that certain valley? What do all those snakes eat? Don't some snakes die every year? Aren't some snakes killed by predators such as hawks and mongeese (mongooses?) How much territory does a snake need to survive? What is the life expectancy of a snake? Some Internet sites say the life expectancy of a snake is 10 to 40 years depending on the species.

Suppose the valley is 2 kilometers long by 1 kilometer wide. The area is 2 square kilometers or (2000 meter $\times$ 1000 meters) = 2,000,000 square meters. Let's calculate the snake population density as Gaia turns and year follows year.

Number of snakes = 6×2^n		Snake population density
Year	Number of snakes	Number of snakes per square meter
4	96	0.000048
5	192	0.000096
10	6144	0.00372
20	6,291,456	3
21	12,582,912	6
22	25,165,824	13
30	6,442,450,944	3221
40	Your turn. please calculate.	Your turn. Please calculate

The land area of the Earth is about 149 trillion square meters (149×10^{12} square meters). According to this model of population growth, in how many years will there be 1 snake per square meter of Earth's land area?

Answer: about 44.5 years.

Your Turn

According to this model of snake population growth (number of snakes = 6×2^n), in how many years will there be 1 snake per square centimeter of Earth's land area?

Wow! Knee deep in snakes. Let's all work on our snake-charming skills!

Alas, how many students every year are taught bad science by test questions such as this one?

A Note on The Science of NFL Football | TOC

We posted this on the Oregon Council of Mathematics (OCTM) listserv on 2011-09-25. It was published in *The Oregon Mathematics Teacher* (TOMT), Nov/Dec 2011. Thanks, Jackie!

Ahoy Mathemagicians,

As suggested by this week's MathNexus, we watched The Science of NFL: Pythagorean Theorem video.

> Watch this video at http://www.nbclearn.com/portal/site/learn/nfl/cuecard/51220/
> MathNexus link: http://mathnexus.wwu.edu/Archive/websites/detail.asp?ID=237

Amusing. A receiver catches a football and runs 40 yards along the side of a triangle perpendicular to the goal line. A defender is initially 30 yards to the receiver's right (at a right angle) and has to run 50 yards along the hypotenuse of the triangle in the same time in order to tackle the ball carrier.

Suppose the ball carrier is a wide receiver. Wide receivers are **fast**! Typical time for a wide receiver to run 40 yards is 4.3 to 4.5 seconds.

The defender who has to run 50 yards to make the tackle is probably a corner back, a safety, or a linebacker. Corner backs are fast too. Linebackers maybe not so fast. Don't know about safeties.

Assume that the defender is as fast as the ball carrier. When the ball carrier has run 40 yards along the side of the right triangle perpendicular to the goal line, the defender has run 40 yards along the hypotenuse and, alas, is still 10 yards from the ball carrier. Does he make the tackle?

The defender runs the last 10 yards along the hypotenuse, but the ball carrier is no longer there. He has run another 10 yards towards the goal line.

Suppose the ball carrier runs 40 yards in 4.3 seconds. Speed = 9.30 yd/s.

To make the tackle, the defender must run 50 yards in 4.3 seconds.

Here are two handy Internet sites about 40-yard and 100-meter dash times:

> 40-yard dash - Wikipedia http://en.wikipedia.org/wiki/40-yard_dash
> 100 metres- Wikipedia http://en.wikipedia.org/wiki/100_metres

Your Turn: Calculate the defender's speed in yards per second.

How does the speed you calculated compare to Ursain Bolt's world record speed in the 100 meters? Bolt's 100-meter time is 9.58 seconds.

Your Turn. Calculate Bolt's speed in meter per second.

In order to compare Bolt's speed in meters per second to the defender's speed in yards per second, you must convert meters per second to yards per second or convert yards per second to meters per second.

1 yard = 0.9144 meter [exactly]
1 meter = 1.0936 yard [approximately]

In order to make the tackle, would the defender have to run faster than Bolt's world-record 100-meter time? If yes, how much faster? What percent faster? Et cetera, et cetera.

When the defender has run 50 yards along the hypotenuse and reached the place where the ball carrier had passed moments ago, by how much time was he late?

Suppose that the receiver was already running toward the goal line when he caught the football. The defender was probably not at that time running along the hypotenuse, but had to react to the catch and start from a standstill. It will take him a few yards to get up to speed. His average speed for the entire 50 yards will be less than you calculated above.

How did the defender know that he would have to run along the 50-yard hypotenuse of the 30-40-50 right triangle? How long did it take him to do this calculation before he started running? How far did the ball carrier run while the defender was calculating?

Well, we could go on, and you can probably think of questions that didn't occur to us.

What do you think that your students might learn by watching this video?

Round-Robin Tournament SAT Practice Test Question | TOC

How do you prepare to take the SAT test? One way is to take online practice tests. We found several practice tests, including one at Kaplan's Test Prep site:

> http://216.154.212.161/KaplanQuizzes/showQuiz.jsp?TID=SATMPT1.

On 2012-02-24 (Friday), Question #22 of the Kaplan test looked like this:

At a basketball tournament involving 8 teams, each team played 4 games with each of the other teams. How many games were played at this tournament?

> A. 64
> B. 98
> C. 112
> D. 128
> E. 224

We selected C. 112 as our answer. Later we looked at Kaplan's answer. Kaplan said that the correct answer is D. 128.

We disagreed, so we filled out a registration form and obtained Kaplan's explanation of its answer, paraphrased here:

> Difficulty: High
>
> Strategic Advice: If each team played 4 games with each of the other teams, then each team played (8)(4) = 32 games. Use this number to determine how many total games were played.
>
> 32 games were played, and each game involved 2 teams. Therefore: 32(8/2) = 128 games were played.

Interesting. There were eight teams and, as stated in the question, each team played 4 games with each **other team**. [Bold added for emphasis.]

Seems to us that each team played 4 games with each of seven other teams. Each team played 28 games. Apparently Kaplan thinks that each team also played itself. It would be fun to watch. We modified Kaplan's expression: 28(8/2) = 112 games were played.

We have never heard of a basketball tournament in which each team played each other team four times. In some sports such as baseball and basketball, there are elimination rounds with pairs of teams playing a short series, but that is not at all like the Kaplan problem.

We have heard of round-robin tournaments in which each team plays each other team **once**, and found many examples on the Internet. We also learned that there are double round-robin tournaments in which each team plays each other team **twice**. After much research, we are quite confident that, in the real world, there are no round-robin tournaments in which each team plays each other team four times. Kaplan's question is unrelated to the real world as – alas – are too many math test questions.

We think that a real-world round-robin tournament question in which each team plays each other team once is an excellent question. On the Net, we found such round-robin tournaments for archery, bowling, basketball, billiards (and pocket billiards, aka pool), chess, cricket, curling, darts, hockey, ping pong, poker, rugby, skeet, soccer, swimming, and tennis.

Boggle, boggle. Why does Kaplan confound an excellent real-world question by turning it into an unreal-world question? The number of games played in an 8-team round-robin tournament in which each team plays each team once is an excellent question. Asking how many games are played in a round-robin tournament in which each team plays each other team four times is silly, unreal-world busy work.

The number of games played in a round-robin tournament in which each team plays each other team once is a **triangular number**. A splendid teaching opportunity.

Two teams: A and B. One game is played. Fanfare: 1 is the **first triangular number**.

	Team A	Team B
Team A		1
Team B		

Three teams: A, B, and C. Three games are played. 3 is the **second triangular number**.

	Team A	Team B	Team C
Team A		1	1
Team B			1
Team C			

Four teams: A, B, C, and D. Six games are played. 6 is the **third triangular number**.

	Team A	Team B	Team C	Team D
Team A		1	1	1
Team B			1	1
Team C				1
Team D				

Eight teams: A, B, C, D, E, F, G, and H. 28 games are played. 28 is the **7th triangular number**.

	Team A	Team B	Team C	Team D	Team E	Team F	Team G	Team H
Team A		1	1	1	1	1	1	1
Team B			1	1	1	1	1	1
Team C				1	1	1	1	1
Team D					1	1	1	1
Team E						1	1	1
Team F							1	1
Team G								1
Team H								

Let there be n teams. The number of games played is the triangular number $n - 1$.

number of games played = $(n - 1)(n) / 2$

Round-robin tournaments: Internet resources

Round-robin tournament – Wikipedia http://en.wikipedia.org/wiki/Round-robin_tournament

"In a single round-robin schedule, each participant plays every other participant once. If each participant plays all others twice, this is frequently called a double round-robin. The term is rarely used when all participants play one another more than twice."

Round-robin draws for curling from 5 teams to 20 teams:

http://media.curling.ca/contentImages/File/Business%20of%20Curling/Operations%20Manual/Chapter%20N%20-%20Draws(1).pdf

Round-Robin Tournament Brackets http://www.printyourbrackets.com/roundrobin.html

Measure the Height of a Tall Object | TOC

We sent this information by email to students in Professor Richard Zimmer's Mars Society course at Sonoma State University, Fall 2003. In another SSU course, two of the students had been assigned the task of measuring the heights of tall objects on the SSU campus.

Subject: Mars Backpack 2003-10-22: Measure the heights of tall objects

Ahoy Margie and Margie,

We loaned you a trundle wheel to use in your project to measure the heights of tall objects on campus. Didn't know at the time that there are two Margies in the Mars Society class, so we're sending this email to both of you, and copying other Mars Trekkers. Here are ways to measure the tall object's height using various tools and toys.

We got the trundle wheel from Nasco
http://www.enasco.com/product/TB24686T.

45-45-90 Degree Triangle Method

Make a large 45-45-90 degree right triangle. This triangle has two legs and one hypotenuse. The two legs are **congruent** – they have the same length.

1. Hold the triangle near your eye with the horizontal leg adjacent to your eye and the vertical leg opposite your eye.
2. While sighting along the hypotenuse, walk backward until the hypotenuse is in line with the top of the tall object. Or until you fall into a ditch, trip over a bench, et cetera, et cetera.
3. The height of the tall object is equal to your horizontal distance from the base of the tall object plus the height of your eye from the ground.

This method is related to the Native American method described at

> RNZIH - Notable Trees New Zealand - Tree Height Calculation Methods
> http://www.rnzih.org.nz/pages/notable5.html

Clinometer Method

Make and use a **clinometer**, or get a ready-made clinometer.

> We like the clinometer from Nasco shown over there →
> http://www.enasco.com/product/TB15983M

How to use or make a clinometer is described at

> Using a clinometer to measure tree height
> http://www.offwell.free-online.co.uk/newpage2.htm
>
> RNZIH - Notable Trees New Zealand - Tree Height Calculation Methods - 3
> http://www.rnzih.org.nz/pages/notable7.html#vertical

What about errors? Suppose you measure the horizontal distance to the base of the tall object very accurately. Measuring the angle accurately may be more difficult. How can you estimate the height error if the angle error is off by, say, 1 degree? You can usually choose the horizontal distance so that the angle is between 0 degrees and 90 degrees. What sighting angle will minimize the error in the height due to an angle error of plus or minus 1 degree?

More Methods from the Internet

Learn about The Shadow Method, pencil method, angle of elevation method, and fixed angle of elevation methods at

> Measure the Height of a Tree http://www.wikihow.com/Measure-the-Height-of-a-Tree

Methods by Bob & George

The Defy, then Use Gravity Method. Fly, walk, teleport, or otherwise go to the top of the tall object carrying a stopwatch and a smooth dense solid sphere. Drop the sphere and time its descent to the base of the tall object. Use an appropriate formula to calculate the distance it dropped which, of course, is the height from which you dropped it.

The What Goes Up Must Come Down Method. Launch a projectile vertically from the base of the tall object. Adjust the launch velocity so that the projectile exactly reaches the top of the tall object, then turns around and returns to the launch point. Station an observer sufficiently far from the launch point so that parallax is not a problem. The observer and the launcher will communicate to adjust the launch velocity using appropriate technology, such as smoke signals, arm and hand signals, other body language, cell phones, walkie-talkies, et cetera, et cetera. Use the total up and down time and an appropriate formula to calculate the height.

The Hot-Air or Helium Balloon Method. Attach a long string to a hot-air- or helium-filled balloon. Allow the balloon to rise until its top is level with the top of the tall object. Mark the string on your end. Pull the balloon back down and measure the distance from your mark to the top of the balloon. To determine when the top of the balloon is at the top of the tall object, station an observer at an appropriate place with appropriate communication technology.

The Kite Method. See the Hot-Air or Helium Balloon Method. Use a kite instead of a balloon. Adjust methodology appropriately.

The Statistical Survey Method. In the vicinity of the tall object, ask *n* people to estimate the height of the tall object – in meters, please. You choose the value of *n*. Calculate the arithmetic mean and other statistics of the data, including the standard error. Meditate on the cosmic meaning of the result.

Reality expands to fill the available fantasies.

Reprise

We sent a copy of the above to David Moursund http://iae-pedia.org/David_Moursund. He replied with a link to

http://www.snopes.com/college/exam/barometer.asp

This Internet site tells the tale of a physics professor who gives his students the task of measuring the height of a tall building by using a barometer. The professor had in mind that students would use the difference in atmospheric pressure at the base and top of the building to calculate the building's height.

One student (the article says it was Niels Bohr) was very creative and suggested several ways to solve the problem, none of which was the way intended by the professor. Read them at http://www.snopes.com/college/exam/barometer.asp.

We replied thusly:

Ahoy Dave,

Thanks! We loved Niels Bohr's methods.

Our mentees were not allowed to go to the top of the tall object, so they could not use the lowering of the barometer on a string method. Some of the tall objects were poles so getting to the top of them would have been difficult.

For objects a few meters to a few tens of meters tall, we suspect that the difference in barometric pressure would be very small and hard to measure. The difference might be less that the accuracy of the barometer.

A high-quality aneroid barometer might have an accuracy of about ± 1 mb (millibar), or about ± 0.1 kPa (1 Pa = 1 pascal = 1 newton per meter). Standard pressure at sea level is 1013 mb = 101.3 kPa.

See http://starpath.com/catalog/accessories/1872f.htm

You can calculate atmospheric pressure as measured by a barometer at

http://hyperphysics.phy-astr.gsu.edu/hbase/Kinetic/barfor.html

Subtract the pressure at various heights above sea level from the sea-level pressure of 101.3 kPa.

Height above sea level	Pressure	Difference from sea-level pressure
0 m	101.3 kPa	0
10 m	101.2 kPa	101.3 kPa – 101.2 kPa = 0.1 kPa
20 m	101.0 kPa	101.3 kPa – 101.0 kPa = 0.3 kPa
30 m	100.9 kPa	101.3 kPa – 100.9 kPa = 0.4 kPa
100 m	100.0 kPa	101.3 kPa – 100.0 kPa = 1.3 kPa

These pressure differences are of the order of the accuracy of the barometer, so can't be relied on.

A quartz barometer is more accurate, but expensive. See http://www.paroscientific.com/resquartzbaro.htm.

The student also suggested using the difference in acceleration due to gravity at the bottom and top of the tall building. We paraphrase:

Tie a string to the barometer and use it as a pendulum. Find the period of the pendulum at the bottom and top of the tall building. The period depends on the length of the string L and the gravitational acceleration g. Formula →

$$T = 2\pi\sqrt{\frac{L}{g}}$$

We comment on this method. The difference in gravitational acceleration (g) at the bottom and top of an object a few meters to a few tens of meters tall is very small. At sea level, g at the top of a 100-meter tall object is about 0.99997 times g at sea level. We suspect that Bohr and others did not have instruments sufficiently accurate to measure that difference.

Hey! Did you go to http://www.snopes.com/college/exam/barometer.asp and read about the clever solutions to the problem?

Win a Bag of Money | TOC

You are a small winner on a TV game show. Your prize is a bag of pennies, but there is a catch.

You may have as many pennies as you can carry 1 kilometer in 1000 seconds or less in one trip. [Average speed greater than or equal to 1 meter per second.]

1 meter per second (m/s) = 3.6 kilometers per hour (km/h)
1 meter per second (m/s) = 2.237 miles per hour (mi/h or mph).

The pennies are all dated 1983 or later, so the mass of each penny is 2.5 grams. [The mass of a penny dated earlier than 1982 is 3.1 grams. The mass of a 1982 penny is 2.5 grams or 3.1 grams. See Penny – Wikipedia http://en.wikipedia.org/wiki/Penny_(United_States_coin).]

How many pennies will you take as your prize?

What is the value in dollars of your prize?

What is the mass of your prize in grams and kilograms?

What is the weight of your prize in newtons (metric) and pounds?

See Weight – Wikipedia http://en.wikipedia.org/wiki/Weight

We like this problem. Why we like it:

It intertwingles math and science.

It doesn't have a single "right" answer.

The answer is personal. How much can **I** carry 1 km in 1000 s?

Relates to grade level. Older, bigger, stronger kids can carry more.

The ability of a person to carry a bag of money 1 kilometer in 1000 seconds or less depends on the person's strength, stamina, and other **life science** attributes. We suspect that students will be overly optimistic about how much they can carry 1 km in 1000 s.

Think about carrying a bag of money versus carrying the money in a backpack. Easier using the backpack, we think. Calculate the volume of a backpack given its length, width, and height. How Many pennies will it hold and how much will they weigh?

Model a penny as a circular disk with diameter 19.05 mm (1.905 cm) in diameter and thickness 1.55 mm (0.155 cm). Calculate the volume:

$$V = \pi r^2 h = \pi(1.905\,\text{cm}/2)^2(0.155\,\text{cm}) = 0.442\,\text{cm}^3$$

Penny image: http://en.wikipedia.org/wiki/Penny_(United_States_coin)

A typical day pack capacity is 32,000 cubic centimeters (1953 cubic inches). Wild stab: Suppose that 75% of the volume of the day pack is occupied by pennies and the other 25% is air space between pennies.

Volume of pennies = 0.75 (32,000 cm^3) = 24,000 cm^3

Number of pennies = (24,000 cm^3) / (0.442 cm^3 per penny) = 54,298 pennies

Mass of the pennies = (54,298 pennies) (2.5 grams per penny) = 135,745 grams

Oops! 135,745 grams is equal to about 136 kilograms, too much for Bob to carry. [For the metric challenged, 136 kilograms has a weight of 300 pounds.]

How many pennies can **you** carry in the backpack 1 kilometer in 1000 seconds or less? What percent of the volume of the day pack is occupied by your pennies?

Bob would rather win a bag of dollar bills. The mass of a dollar bill is 1 gram.

1000 $1 bills: 1000 g = 1 kg [about 2.2 lb]

10,000 $1 bills: 10,000 g = 10 kg [about 22 lb]

Bob thinks he can carry 10,000 $1 bills 1 kilometer in 1000 seconds or less.

Winners might win pennies, nickels, dimes, quarters, half dollars, Presidential $1 coins, or Native American $1 coins. Here is a table of masses, diameters, and thicknesses of these coins.

US Mint Coin Specifications

http://www.usmint.gov/about_the_mint/index.cfm?action=coin_specifications

Coin	Value ($)	Mass (g)	Diameter cm)	Thickness (cm)
cent (penny)	0.01	2.500	1.905	0.155
nickel	0.05	5.000	2.121	0.195
dime	0.10	2.268	1.791	0.135
quarter	0.25	5.670	2.426	0.175
half dollar	0.50	11.340	3.061	0.215
Presidential $1	1.00	8.100	2.649	0.200
Native America $1	1.00	8.100	2.649	0.200

How much would you pay a Sherpa to carry your bag of money 1 kilometer in 1000 seconds or less?

Solar System Necklace | TOC

Bob's son Karl sent a link to a site that sells a solar system necklace kit for $30.

Go to http://www.makershed.com/Solar_System_Necklace_Kit_p/cgb2.htm

The kit contains beads, a hunk of wire, and a magnetic clasp. It looks like the small black beads provide the spacing between planets. Each black bead is 20 million miles.

20 million miles = 32.2 million kilometers = 0.215 AU (astronomical unit).

Distances of planets from the Sun and distances between planets:

Mercury: 0.39 AU, 2 beads from the Sun

Venus: 0.72 AU, 3 beads from the Sun, 1 bead from Mercury

Earth: 1.00 AU, 5 beads from the Sun, 2 beads from Venus.

Mars: 1.52 AU, 7 beads from the Sun, 2 beads from Earth

Asteroids: 2.50 AU, 12 beads from the Sun, 5 beads from Mars

Jupiter: 5.20 AU, 24 beads from the Sun, 12 beads from the asteroids

Saturn: 9.58 AU, 45 beads from the Sun, 21 beads from Jupiter

Uranus: 19.2 AU, 89 beads from the Sun, 44 beads from Saturn

Neptune: 30.0 AU, 140 beads from the Sun, 51 beads from Uranus

Pluto: 39.5 AU, 184 beads from the Sun, 44 beads from Neptune

Total number of spacer beads = 184.

The length of the necklace is 97 centimeters (970 millimeters).

Ignoring the larger beads that represent planets, the small black beads are about

970 mm / 184 = 5.3 mm in diameter.

Call it 5 mm = 0.5 cm, half the width of your little fingernail.

We looked for bead sizes on the Net. 5 mm is a standard bead size.

The scale is 5 mm : 0.215 AU or 5 mm : 32.2 million kilometers or 1 mm : 6.43 million kilometers.

The planet beads can't be to the same scale. Planets would be too small to see without a strong magnifier. Earth's bead would be about 0.002 millimeters in diameter and Jupiter's bead would be about 0.02 mm in diameter. Tiny!

Hey! The necklace costs $30. Using the above data, you can buy some beads ($1? $2?), a bit of wire or string, a clasp, and make the necklace. We think this is an excellent classroom team project, especially if your students do the calculations that we did above.

Coconut-Carrying Swallows | TOC

We don't recall where we found this math problem and we haven't found it on the Net. Here's the problem:

> The average flight velocity of an African Swallow is 17/3 that of the European Swallow. An African Swallow and two European Swallows collaborate in transporting 50 coconuts a distance of 6 km. If the average velocity of a European Swallow is 9 km/hour while carrying a coconut and 18 km/hour while unburdened, how many minutes will the job require? (i.e. when will the Swallow carrying the last coconut pass the 6 km mark?)

Here is a reality check on some of the supposed real world components of this problem:

Wikipedia: the mass of a swallow is 10 to 60 grams and an unburdened swallow usually travels at 30 to 40 km/h.http://en.wikipedia.org/wiki/Swallow

Wikipedia: a "full grown" coconut weighs about 1.44 kilogram (1,440 grams). http://en.wikipedia.org/wiki/Coconut

> How does a big swallow (60 grams) carry a 1,440 gram coconut?
>
> 1440g / 60g = 24. Can a swallow fly and carry 24 times its body mass?

The problem states that an unburdened swallow flies 18 km/h. But Wikipedia says that typical speed while foraging is 30 to 40 km/h.

The problem states that an unburdened European swallow flies 18 km/h and an African swallow flies 17/3 times that fast or 102 km/h. Does that sound reasonable?

Is this problem reasonable? Does it teach good science? Does it use accurate data? Should we use it with students? If we pose the problem to students, should we discuss the accuracy of the data and the credibility of the problem?

Use FOIL to Multiply 2-Digit Numbers | TOC

Have you ever thought about using **FOIL** (**F**irst **O**uter **I**nner **L**ast) to multiply 2-digit numbers?

Here are examples using FOIL to multiply two **binomials** in the variable *x*, and then using FOIL to multiply two 2-digit numbers written as binomials where *x* is replaced by 10. The two examples illustrate the **commutative property of multiplication**: $(x + 1)(x + 2) = (x + 2)(x + 1)$ and (10 + 1)(10 + 2) = (10 + 2)(10 + 1).

$(x+1)(x+2)=x^2+2x+1x+1\cdot 2=x^2+3x+2$ 11 × 12 = (10 + 1)(10 + 2). Looks like $(x + 1)(x + 2)$ with *x* replaced by 10. $(10+1)(10+2)=10^2+2\cdot 10+1\cdot 10+1\cdot 2=10^2+3\cdot 10+2=100+30+2=132$
$(x+2)(x+1)=x^2+1x+2x+2\cdot 1=x^2+3x+2$ 12 × 11 = (10 + 2)(10 + 1). Looks like $(x + 2)(x + 1)$ with *x* replaced by 10. $(10+2)(10+1)=10^2+1\cdot 10+2\cdot 10+2\cdot 1=10^2+3\cdot 10+2=100+30+2=132$

Abracadaba! FOIL worked with the binomials $x+1$ and $x+2$, and FOIL also worked with the numbers 11 and 12 written as binomials 10 + 1 and 10 + 2.

$$(x+1)(x+2)$$
$$(10+1)(10+2)$$

Here are two more examples, again illustrating the **commutative property of multiplication**.

$(x+2)(x+3)=x^2+3x+2x+2\cdot 3=x^2+5x+6$ 12 × 13 = (10 + 2)(10 + 3). Looks like $(x + 2)(x + 3)$ with *x* replaced by 10. $(10+2)(10+3)=10^2+3\cdot 10+2\cdot 10+2\cdot 3=10^2+5\cdot 10+6=100+50+6=156$
$(x+3)(x+2)=x^2+2x+3x+3\cdot 2=x^2+5x+6$ 13 × 12 = (10 + 3)(10 + 2). Looks like $(x + 3)(x + 2)$ with *x* replaced by 10. $(10+3)(10+2)=10^2+2\cdot 10+3\cdot 10+3\cdot 2=10^2+5\cdot 10+6=100+50+6=156$

Alakazam! FOIL worked with the binomials $x+2$ and $x+3$, and FOIL also worked with the numbers 12 and 13 written as binomials 10 + 2 and 10 + 3.

$$(x+2)(x+3)$$
$$(10+2)(10+3)$$

More products of binomials in *x* and numbers written as binomials
$(x+2)(x+5) = x^2 + 5x + 2x + 2 \cdot 5 = x^2 + 7x + 10$ $12 \times 15 = (10 + 2)(10 + 5)$. Looks like $(x + 2)(x + 5)$ with x replaced by 10. $(10+2)(10+5) = 10^2 + 5 \cdot 10 + 2 \cdot 10 + 2 \cdot 5 = 10^2 + 7 \cdot 10 + 10 = 100 + 70 + 10 = 180$
$(x+3)(x+5) = x^2 + 5x + 3x + 3 \cdot 5 = x^2 + 8x + 15$ $13 \times 15 = (10 + 3)(10 + 5)$. Looks like $(x + 3)(x + 5)$ with x replaced by 10. $(10+3)(10+5) = 10^2 + 5 \cdot 10 + 3 \cdot 10 + 3 \cdot 5 = 10^2 + 8 \cdot 10 + 15 = 100 + 80 + 15 = 195$
$(x+8)(x+9) = x^2 + 9x + 8x + 8 \cdot 9 = x^2 + 17x + 72$ $18 \times 19 = (10 + 8)(10 + 9)$. Looks like $(x + 8)(x + 9)$ with x replaced by 10. $(10+8)(10+9) = 10^2 + 9 \cdot 10 + 8 \cdot 10 + 8 \cdot 9 = 10^2 + 17 \cdot 10 + 72 = 100 + 170 + 72 = 342$

Squares of binomials in *x* and squares of numbers written as binomials
$(x+1)^2 = (x+1)(x+1) = x^2 + 1x + 1x + 1 \cdot 1 = x^2 + 2x + 1$ $11 \times 11 = (10 + 1)(10 + 1)$. Looks like $(x + 1)(x + 1)$ with x replaced by 10. $(10+1)(10+1) = 10^2 + 1 \cdot 10 + 1 \cdot 10 + 1 \cdot 1 = 10^2 + 2 \cdot 10 + 1 = 100 + 20 + 1 = 121$
$(x+6)^2 = (x+6)(x+6) = x^2 + 6x + 6x + 6 \cdot 6 = x^2 + 12x + 36$ $16 \times 16 = (10 + 6)(10 + 6)$. Looks like $(x + 6)(x + 6)$ with x replaced by 10. $(10+6)(10+6) = 10^2 + 6 \cdot 10 + 6 \cdot 10 + 6 \cdot 6 = 10^2 + 12 \cdot 10 + 10 = 100 + 120 + 36 = 256$
$(x+9)^2 = (x+9)(x+9) = x^2 + 9x + 9x + 9 \cdot 9 = x^2 + 18x + 81$ $19 \times 19 = (10 + 9)(10 + 9)$. Looks like $(x + 9)(x + 9)$ with x replaced by 10. $(10+9)(10+9) = 10^2 + 9 \cdot 10 + 9 \cdot 10 + 9 \cdot 9 = 10^2 + 18 \cdot 10 + 81 = 100 + 180 + 81 = 361$

You can write a binomial in the variable *x* as $ax + b$, where *a* and *b* are real numbers.

You can write a 2-digit natural number as $a \cdot 10 + b$, where *a* = 1 to 9 and *b* = 0 to 9.

More products of binomials in *x* and numbers written as binomials
$(2x+3)(4x+5) = 8x^2 + 10x + 12x + 3 \cdot 5 = 8x^2 + 22x + 15$ $(2 \cdot 10 + 3)(4 \cdot 10 + 5) = 1035$. Looks like (2*x* + 3)(4*x* + 5) with *x* replaced by 10. $(2 \cdot 10 + 3)(4 \cdot 10 + 5) = 8 \cdot 10^2 + 10 \cdot 10 + 12 \cdot 10 + 15 = 8 \cdot 10^2 + 22 \cdot 10 + 3 \cdot 5 = 800 + 220 + 15 = 1035$
$(3x+2)(2x+3) = 6x^2 + 9x + 4x + 2 \cdot 3 = 6x^2 + 13x + 15$ $(3 \cdot 10 + 2)(2 \cdot 10 + 3) = 736$. Looks like (3*x* + 2)(2*x* + 3) with *x* replaced by 10. $(3 \cdot 10 + 2)(2 \cdot 10 + 3) = 6 \cdot 10^2 + 9 \cdot 10 + 4 \cdot 10 + 2 \cdot 3 = 6 \cdot 10^2 + 13 \cdot 10 + 6 = 600 + 130 + 6 = 736$
$(x+2)(3x+5) = 3x^2 + 5x + 6x + 2 \cdot 5 = 3x^2 + 11x + 10$ $(10+2)(3 \cdot 10 + 5) = 420$. Looks like (*x* + 2)(3*x* + 5) with *x* replaced by 10. $(10+2)(3 \cdot 10 + 5) = 3 \cdot 10^2 + 5 \cdot 10 + 6 \cdot 10 + 2 \cdot 5 = 3 \cdot 10^2 + 11 \cdot 10 + 10 = 300 + 110 + 10 = 420$
$(3x+7)(x+6) = 3x^2 + 18x + 7x + 7 \cdot 6 = 3x^2 + 25x + 42$ $(3 \cdot 10 + 7)(10+6) = 592$. Looks like (3*x* + 7)(*x* + 6) with *x* replaced by 10. $(3 \cdot 10 + 7)(10+6) = 3 \cdot 10^2 + 18 \cdot 10 + 7 \cdot 10 + 7 \cdot 6 = 3 \cdot 10^2 + 25 \cdot 10 + 42 = 300 + 250 + 42 = 592$
$(3x+7)(x+6) = 3x^2 + 18x + 7x + 7 \cdot 6 = 3x^2 + 25x + 42$ $(3 \cdot 10 + 7)(10+6) = 592$. Looks like (3*x* + 7)(*x* + 6) with *x* replaced by 10. $(3 \cdot 10 + 7)(10+6) = 3 \cdot 10^2 + 18 \cdot 10 + 7 \cdot 10 + 7 \cdot 6 = 3 \cdot 10^2 + 25 \cdot 10 + 42 = 300 + 250 + 42 = 592$
$(9x+9)(5x+5) = 45x^2 + 45x + 45x + 9 \cdot 5 = 45x^2 + 90x + 45$ $(9 \cdot 10 + 9)(5 \cdot 10 + 5) = 5445$. Looks like (9*x* + 9)(5*x* + 5) with *x* replaced by 10. $(9 \cdot 10 + 9)(5 \cdot 10 + 5) = 45 \cdot 10^2 + 45 \cdot 10 + 45 \cdot 10 + 9 \cdot 5 = 45 \cdot 10^2 + 45 \cdot 10 + 45 = 4500 + 450 + 45 = 5445$

Squares of binomials in *x* and squares of numbers written as binomials
$(2x+3)^2 = (2x+1)(2x+1) = 4x^2 + 2x + 2x + 1\cdot 1 = 4x^2 + 4x + 1$ $23^2 = (2\cdot 10+3)^2 = (2\cdot 10+3)(2\cdot 10+3) = 529$. Looks like (2$x$ + 3)(2x + 3) with x replaced by 10. $(2\cdot 10+3)(2\cdot 10+3) = 4\cdot 10^2 + 6\cdot 10 + 6\cdot 10 + 3\cdot 3 = 4\cdot 10^2 + 12\cdot 10 + 9 = 400 + 120 + 9 = 529$
$(3x+6)^2 = (3x+6)(3x+6) = 9x^2 + 18x + 18x + 6\cdot 6 = 9x^2 + 36x + 36$ $36^2 = (3\cdot 10+6)^2 = (3\cdot 10+6)(3\cdot 10+6) = 1296$. Looks like ($x$ + 6)(x + 6) with x replaced by 10. $(3\cdot 10+6)(3\cdot 10+6) = 9\cdot 10^2 + 18\cdot 10 + 18\cdot 10 + 6\cdot 6 = 9\cdot 10^2 + 36\cdot 10 + 36 = 900 + 360 + 36 = 1296$
$(3x+3)^2 = (3x+3)(3x+3) = 9x^2 + 9x + 9x + 3\cdot 3 = 9x^2 + 18x + 9$ $33^2 = (3\cdot 10+3)^2 = (3\cdot 10+3)(3\cdot 10+3) = 1089$. [(3$x$ + 3)(3x + 3) with x replaced by 10.] $(3\cdot 10+3)(3\cdot 10+3) = 9\cdot 10^2 + 9\cdot 10 + 9\cdot 10 + 3\cdot 3 = 9\cdot 10^2 + 18\cdot 10 + 9 = 900 + 180 + 9 = 1089$
$(9x+9)^2 = (9x+9)(9x+9) = 81x^2 + 18x + 18x + 9\cdot 9 = 81x^2 + 36x + 81$ $99^2 = (9\cdot 10+9)^2 = (9\cdot 10+9)(9\cdot 10+9) = 9801$. [(9$x$ + 9)(9x + 9) with x replaced by 10. $(9\cdot 10+9)(9\cdot 10+9) = 81\cdot 10^2 + 81\cdot 10 + 81\cdot 10 + 9\cdot 9 = 81\cdot 10^2 + 18\cdot 10 + 81 = 8100 + 1620 + 81 = 9801$

Well, that's enough FOILing around for today.

Nine to the Nine to the Nine | TOC

The MathNexus story about Stu Dent and Poly Dent is much fun.

When My Dear Aunt Sally Did Not Help!

http://mathnexus.wwu.edu/Archive/problem/detail.asp?ID=255

The problem: Evaluate 9^{9^9}

Is that 9 to the (9 to the 9), or is it (9 to the 9) to the 9? $9^{\left(9^9\right)}$ or $\left(9^9\right)^9$?

Without parentheses, the usual order of operations is to do the exponentiations in left to right order. First calculate 9^9, and then calculate the 9th power of that result. That's the way the TI-94 graphing calculator does it.

TI-84: 9^9^9 = 1.966270505E77. This value is rounded to 10 digits.

TI-84: (9^9)^9 produces the same result, 1.966270505E77.

The E77 tells you that the exact value has 77 digits to the right of the decimal point. The number has 78 digits.

WinCalc to the rescue. Download WinCalc at Peanut Software http://math.exeter.edu/rparris/. You can use WinCalc to do exact arithmetic with hundreds of digits, thousands of digits, even more digits.

Beware. WinCalc does not use the left to right order of operations for this problem. If you enter 9^9^9, it calculates 9^9 = 387420489, and then tries to calculate 9^387420489. This results in an "out of bounds" message.

So try (9^9)^9. WinCalc calculates the exact answer, shown here:

196627050475552913618075908526912116283103450944214766927315415537966391196809

Count the digits. Yep, 78 digits.

To make the digit-counting task easier, we rewrote the number in 9-point Arial and put a space every three digits from the right. Put on your reading glasses and count digits:

196 627 050 475 552 913 618 075 908 526 912 116 283 103 450 944 214 766 927 315 415 537 966 391 196 809

Now let's try 9^(9^9) on the TI-84. Oops. `ERR: OVERFLOW`.

We don't know the exact value of 9^(9^9), but we can contrive an upper bound.

9^(9^9) is less than 10^(10^10) = 10^10,000,000,000. If you have the time, you can write 10^(10^10) as 1 followed by 10 billion zeroes.

The exact value of 10^(10^10) has 10 billion and 1 digits.

If you write one zero every second, how long will it take you to write 10,000,000,000 zeroes?

1 day = 86,400 seconds.
1 year = 365 days.
1 year = 365.242 days (more accurate – see http://en.wikipedia.org/wiki/Tropical_year).

We wish you a long life.

N to the N to the N

Evaluate (2^2)^2 and 2^(2^2). Surprise!
Evaluate (3^3)^3 and 3^(3^3).
Evaluate (4^4)^4 and 4^(4^4).
Evaluate (5^5)^5 and 5^(5^5).

Now here's a howdy do! We used WinCalc to evaluate 3^(3^3), 4^(4^4), and 5^(5^5).

WinCalc: 3^(3^3) has **13** digits.
Hey! log (3^(3^3)) = (3^3) log (3). TI-84: (3^3) log (3) = **12**.88....

WinCalc: 4^(4^4) has **155** digits.
Wow! log (4^(4^4)) = (4^4) log (4). TI-84: (4^4) log (4) = **154**.127....

WinCalc: 5^(5^5) has **2185** digits.
Zowie! log (5^(5^5)) = (5^5) log (5). TI-84: (5^5) log (5) = **2184**.281....

TI-84: (6^6) log (6) = 36305.424.... Conjecture: 6^(6^6) has 36,306 digits.

Your Turn.

Your conjecture: How many digits does 7^(7^7) have?

Your conjecture: How many digits does 8^(8^8) have?

Your conjecture: How many digits does 9^(9^9) have? How does this number of digits compare with the number of digits in 10^(10^10)?

Keep on truckin'. 11^(11^11), 12^(12^12), 13^(13^13), et cetera, et cetera.

Eager Eric Wants to be Shorn | TOC

At the Oregon Math Leaders (OML) conference in August 2007, we attended Patty Sandoz's Pizza, Pop, and Problems (P^3) playshop and learned about Eager Eric. Great fun! We kept thinking about Eager Eric, and then wrote this unit.

You can read about Eric the sheep at

http://www.learner.org/channel/courses/learningmath/algebra/session1/part_b/index.html

We loved Eager Eric, and forgive his line-jumping propensity. We especially liked starting with people simulating/emulating Eric and the queue of sheep. Next time, we will apologize as we bump around two "sheep" ahead of us. If you bump ahead in the real world, road rage!

Questions: How long does it take to shear a sheep? How much time does Eric save by line-jumping? Who are these subservient sheep who allow Eric to bump ahead? Why is Eric so eager? We vaguely recall that it was a hot day, so did Eric think that he can become cool by being shorn? Is Eric an adolescent who confuses being cool (thermal) with **being cool**?

Oops! What does Eric do if a sheep objects to his jumping ahead in the line? Worse – what does Eric do if both sheep that he is trying to pass object and prevent him from advancing?

Aha! Much advanced scenario. Flip a coin to determine if Eric can move up. Roll a die to determine how far Eric moves up. But that's another story for another time.

Back to Eric. What say make this a **thought experiment** in which everything happens exactly the way we want it happen. Einstein created relativity theory by means of thought experiments. Quantum theory was created by people using thought experiments. Hmmm...great scientists and mathematicians use thought experiments to create new math and science – are we using thought experiments to teach math and science? We would love it if teachers would pose thought experiments that ... well, that's another story for another time.

EQUAL OPPORTUNITY: ERIC or ERICA

It would be ewesful if we could select Eric (male, ram) or Erica (female, ewe) by, say, flipping the penny that was handed out at the beginning of the activity.

If the coin flip chooses Eager Erica instead of Eager Eric, **then** we can have a **ewe queue**.

Sorry about that. On to Eager Eric.

Korrekshuns to this unit will be greatfully accepted.

Plan A Start with 50 sheep ahead of Eric and work on down the line.

Eager Eric, number of sheep ahead of him, and the number of sheep shorn		
Turn	**Number of sheep ahead of Eric. Each turn 1 sheep is shorn and Eric moves up 2 places in line.**	**Shorn**
0	50	0
1	50 – 1 sheep shorn – 2 sheep passed in line = 50 – 3 = **50 – 3(1)**	1
2	50 – 3(1) – 1 – 2 = 50 – 3(1) – 3 = **50 – 3(2)**	2
3	50 – 3(2) – 1 – 2 = 50 – 3(2) – 3 = **50 – 3(3)**	3
4	50 – 3(3) – 1 – 2 = 50 – 3(3) – 3 = **50 – 3(4)**	4
5	50 – 3(4) – 1 – 2 = 50 – 3(4) – 3 = **50 – 3(5)**	5
...	et cetera, et cetera, ...	
k	$50 - 3(k-1) - 1 - 2 = 50 - 3(k-1) - 3 = \mathbf{50 - 3(k)}$	k
...	et cetera, et cetera,...	
16	50 – 3(15) – 1 – 2 = 50 – 3(15) – 3 = **50 – 3(16) = 2**	16
17	1 sheep is shorn, Eric moves past the 1 remaining sheep ahead of him, and is next to be shorn.*	17

* This assumes that Eric can adjust moving ahead in the line from moving past two sheep to moving past one sheep, if there is only one sheep ahead of him. However, if Eric hesitates when only one sheep is ahead of him and doesn't move up quickly enough, the sheep ahead of him gets shorn and the number shorn before Eric is first in line is 18 instead of 17.

Hey! This looks like the **division algorithm for whole numbers**:

If m and n are whole numbers and $n \neq 0$, then there exist whole numbers q and r such that

$m = nq + r$ where $r < n$. [q is the **quotient** and r is the **remainder** on dividing m by n.]

If $r = 0$, then q is the number of sheep shorn before Eric is first in line.

If $r \neq 0$, then $q + 1$ is the number of sheep shorn before Eric is first in line.

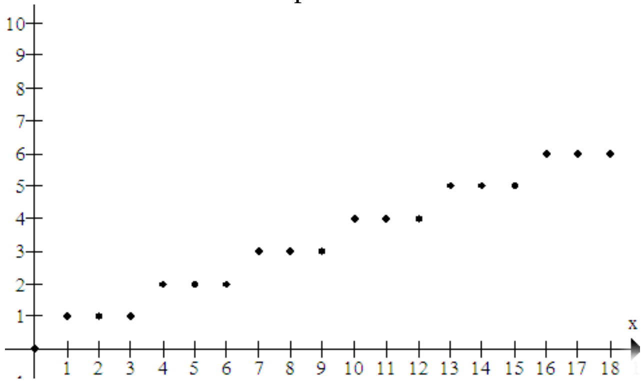
> In this problem: $50 = 3q + r$ where $q = 16$ and $r = 2$. Therefore, $16 + 1 = 17$ is the number of sheep shorn before Eric is first in line.

Auxiliary question: How many sheep did Eric rudely pass on his way to the beginning of the line?

The Creep Up On the Solution Method, inspired by Lynn Bonser

Scenario	**The queue**
1 sheep is ahead of Eric. • **1 sheep is shorn.** • Eric is next. • Number of sheep shorn before Eric = 1.	E S E
2 sheep are ahead of Eric. • **1 sheep is shorn.** • Eric moves past 1 sheep. • Number of sheep shorn before Eric = 1.	E S S E S S E
3 sheep are ahead of Eric. • 1 **sheep is shorn**. • Eric moves past 2 sheep. • Number of sheep shorn before Eric = 1.	E S S S E S S S S E
4 sheep are ahead of Eric. • **1 sheep is shorn**. • Eric moves past 2 sheep. • **1 sheep is shorn**. • Number of sheep shorn before Eric = 2.	E S S S S E S S S S S E S S S E
5 sheep are ahead of Eric. • **1 sheep is shorn**. • Eric moves past 2 sheep. • **1 sheep is shorn**. • Eric moves past 1 sheep. • Number of sheep shorn before Eric = 2.	E S S S S S E S S S S S S E S S S S E S S S S E
6 sheep are ahead of Eric. • **1 sheep is shorn**. • Eric moves up 2 places. • **1 sheep is shorn**. • Eric moves past 2 sheep. • Number of sheep shorn before Eric = 2.	E S S S S S S E S S S S S S S E S S S S S E S S S S S S E

The graph of the number of sheep shorn (*NS*) before Eric is shorn is a function of the number *N* of sheep ahead of Eric initially. The function is a **step function**.

<table>
<tr><td colspan="5">T-table and graph of the number of sheep shorn (NS on the vertical axis) as a function of the number of sheep initially in front of Eric (N on the horizontal axis).</td></tr>
<tr><td>N</td><td>NS</td><td>N</td><td>NS</td><td rowspan="8">⬆ NS = number of sheep shorn before Eric is shorn

N = number of sheep initially ahead of Eric</td></tr>
<tr><td>0</td><td>0</td><td></td><td></td></tr>
<tr><td>1
2
3</td><td>1
1
1</td><td>10
11
12</td><td>4
4
4</td></tr>
<tr><td>4
5
6</td><td>2
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2</td><td>13
14
15</td><td>5
5
5</td></tr>
<tr><td>7
8
9</td><td>3
3
3</td><td>16
17
18</td><td>6
6
6</td></tr>
</table>

The Algebraic Alakazam Method

As the sun rises and sheep queue up to be shorn, suppose that N sheep are between Eager Eric and his heart's desire – to get a Michael Jordan haircut. Each turn, one sheep is shorn and Eric boogies up two places in line. So, tra la, tra la, each time a sheep is shorn, Eric is three places closer to being first in line.

Alakazam! Do whole number division of N by 3 and get a whole number quotient Q and whole number remainder R, where $R = 0$ or $R = 1$ or $R = 2$.

- If $R = 0$, then Q = number of sheep shorn when Eric becomes first in line.
- If $R = 1$, then $Q + 1$ = number of sheep shorn when Eric becomes first in line.
- If $R = 2$, then $Q + 1$ = number of sheep shorn when Eric becomes first in line.

<table>
<tr><td>N = 48

  16 = Q
3|48
  48
   0 = R

R = 0. Therefore:

Sheep shorn = Q = 16</td><td>N = 49

  16 = Q
3|49
  48
   1 = R

R ≠ 0. Therefore:

Sheep shorn = Q + 1 = 17</td><td>N = 50

  16 = Q
3|50
  48
   2 = R

R ≠ 0. Therefore:

Sheep shorn = Q + 1 = 17</td></tr>
</table>

N	***Q***	***R***	**Sheep shorn before Eric**	
0 1 2	0 0 0	0 1 2	0 1 1	
3 4 5	1 1 1	0 1 2	1 2 2	
6 7 8	2 2 2	0 1 2	2 3 3	
9 10 11	3 3 3	0 1 2	3 4 4	
12 13 14	4 4 4	0 1 2	4 5 5	
…	…	…	…	Et cetera, et cetera
48 49 50	16 16 16	0 1 2	16 17 17	

Suppose that Eric becomes even more audacious and moseys up three places in line every time a sheep is shorn. Each time a sheep is shorn, Eric is four places closer to being first in line.

Abracadabra! Do whole number division of N by 4 and get a whole number quotient Q and whole number remainder R, where $R = 0$ or $R = 1$ or $R = 2$ or $R = 3$.

- If $R = 0$, then Q = number of sheep shorn when Eric becomes first in line.
- If $R = 1$, then $Q + 1$ = number of sheep shorn when Eric becomes first in line.
- If $R = 2$, then $Q + 1$ = number of sheep shorn when Eric becomes first in line.
- If $R = 3$, then $Q + 1$ = number of sheep shorn when Eric becomes first in line.

$N = 48$: $48 = 4 \cdot 12 + 0$. $Q = 12$, $R = 0$. Number of sheep shorn before Eric = $Q = 12$.

$N = 49$: $49 = 4 \cdot 12 + 1$. $Q = 12$, $R = 1$. Number of sheep shorn before Eric = $Q + 1 = 13$.

$N = 50$: $50 = 4 \cdot 12 + 2$. $Q = 12$, $R = 2$. Number of sheep shorn before Eric = $Q + 1 = 13$.

$N = 51$: $51 = 4 \cdot 12 + 3$. $Q = 12$, $R = 3$. Number of sheep shorn before Eric = $Q + 1 = 13$.

Are we doing thought experiments? Yes, we are doing thought experiments. Let's think another thought experiment.

Another sheep shearer appears. The two shearers are twins or clones or whatever – they both shear a sheep in exactly the same time. [In a thought experiment, anything is possible.] The two shearers shear two sheep and Eager Eric sidles up three places in line. Each turn, Eric is five places closer to becoming first in line.

Presto modesto! Do whole number division of N by 5 and get a whole number quotient Q and whole number remainder R, where $R = 0$ or $R = 1$ or $R = 2$ or $R = 3$ or $R = 4$.

- If $R = 0$, then Q = number of sheep shorn when Eric becomes first in line.
- If $R = 1$, then $Q + 1$ = number of sheep shorn when Eric becomes first in line.
- If $R = 2$, then $Q + 1$ = number of sheep shorn when Eric becomes first in line.
- If $R = 3$, then $Q + 1$ = number of sheep shorn when Eric becomes first in line.
- If $R = 4$, then $Q + 1$ = number of sheep shorn when Eric becomes first in line.

$N = 50$: $50 = 5 \cdot 10 + 0$. $Q = 10$, $R = 0$. Number of sheep shorn before Eric = $Q = 10$.

$N = 51$: $51 = 5 \cdot 10 + 1$. $Q = 10$, $R = 1$. Number of sheep shorn before Eric = $Q + 1 = 11$.

$N = 52$: $52 = 5 \cdot 10 + 2$. $Q = 10$, $R = 2$. Number of sheep shorn before Eric = $Q + 1 = 11$.

$N = 53$: $53 = 5 \cdot 10 + 3$. $Q = 10$, $R = 3$. Number of sheep shorn before Eric = $Q + 1 = 11$.

$N = 54$: $54 = 5 \cdot 10 + 4$. $Q = 10$, $R = 4$. Number of sheep shorn before Eric = $Q + 1 = 11$.

Generalize, generalize, generalize

Each turn S sheep are shorn and Eric moves up P places in line.

- Each turn Eric is $K = S + P$ places closer to the front of the line.

Aha! Do whole number division of N by K and get a whole number quotient Q and whole number remainder R, where $R = 0, 1, 2, ..., K – 1$.

- If $R = 0$, then Q = number of sheep shorn when Eric becomes first in line.
- If $R \neq 0$, then $Q + 1$ = number of sheep shorn when Eric becomes first in line.

Reality expands to fill the available fantasies.

Bob & George

Roll 2D6 Doubles Game | TOC

How to play:

- Start with your score = 12.
- Roll 2D6 each turn for 12 turns.
- Each turn: If the outcome is not a double, subtract 1 from the score. If the outcome is a double, add the sum of the double to the score. Example: If you roll (3, 3), add 6 to your score.

Bob told Maia & Josh about this game. They played ten games and calculated statistics.

Turn	Roll	Score	Roll	Score	Roll	Score	Roll	Score	Roll	Score
		12		12		12		12		12
1	2, 3	11	3, 6	11	4, 1	11	2, 5	11	4, 5	11
2	3, 2	10	2, 1	10	5, 1	10	5, 1	10	2, 1	10
3	1, 2	9	**3, 3**	16	1, 2	9	**1, 1**	12	5, 4	9
4	2, 5	8	1, 2	15	**4, 4**	17	4, 1	11	3, 4	8
5	1, 2	7	2, 5	14	4, 5	16	3, 1	10	**2, 2**	12
6	6, 5	6	**6, 6**	26	**3, 3**	22	3, 5	9	6, 1	11
7	3, 6	5	**4, 4**	34	5, 4	21	1, 5	8	4, 1	10
8	1, 6	4	5, 2	33	6, 1	20	6, 3	7	1, 5	9
9	6, 3	3	1, 4	32	3, 4	19	4, 6	6	5, 6	8
10	3, 5	2	5, 3	31	4, 1	18	6, 1	5	1, 2	7
11	**3, 3**	8	3, 2	30	4, 1	17	4, 2	4	**3, 3**	13
12	2, 6	**7**	4, 6	**29**	2, 1	**16**	5, 2	**3**	5, 3	**12**

Turn	Roll	Score	Roll	Score	Roll	Score	Roll	Score	Roll	Score
		12		12		12		12		12
1	3, 1	11	6, 4	11	5, 4	11	6, 3	11	3, 6	11
2	5, 3	10	6, 2	10	1, 3	10	4, 5	10	4, 5	10
3	6, 2	9	4, 2	9	1, 3	9	4, 6	9	6, 3	9
4	2, 3	8	3, 1	8	6, 2	8	**5, 5**	19	**1, 1**	11
5	3, 2	7	5, 4	7	5, 2	7	6, 5	18	**2, 2**	15
6	5, 2	6	1, 6	6	6, 5	6	**2, 2**	23	**3, 3**	21
7	5, 4	5	1, 6	5	6, 5	5	2, 5	22	4, 2	20
8	**5, 5**	15	**2, 2**	9	**4, 4**	13	3, 2	21	1, 2	19
9	1, 2	14	4, 1	8	2, 6	12	**3, 3**	27	6, 5	18
10	4, 2	13	5, 1	7	5, 6	11	6, 2	26	5, 4	17
11	3, 6	12	4, 2	6	1, 4	10	1, 4	25	5, 2	16
12	**5, 5**	**22**	6, 3	**5**	**3, 3**	**16**	1, 6	**24**	6, 4	**15**

Maia & Josh Calculate Statistics for the 10 Plays of Bob's Doubles Game

Sum of scores in ten games = 7 + 29 + 16 + 3 + 12 + 22 + 5 + 16 + 24 + 15 = 149

Scores in numerical order: 3 5 7 12 15 16 16 22 24 29

Mean = 14.9 Median = 15.5 Mode = 16 Range = 26

Maia and Josh muse about the game.

We admit that this is not a terrifically exciting game. We like it because it has interesting statistics. You start with 12 points and each turn that you don't roll a double, you lose a point. The game has 12 turns, so if you don't roll a double in any turn, you will end with a score of zero. [M&J are already thinking about variations, disclosed below.]

As you roll 2D6 turn after turn, you lose points and suspense increases. If you roll a double, anxiety turns into a sigh of relief – or perhaps a bit of joy if the double was double 5 or double 6, thus adding a bunch of points to your score.

M&J note that the game involves subtracting 1 from the score or adding the sum of a double to the score, a task that, we think, a lower elementary student can do, perhaps with a base-10 block "calculator."

M&J conjectured that students who played many games might get a "feel" for how often a double might appear. Indeed, if an entire classroom played many games coached by a teacher, they might discover ….

Your Turn

What is the probability of rolling doubles? Complete the table and calculate probabilities.

		Red die					
		1	**2**	**3**	**4**	**5**	**6**
Green die	**1**	(**1**, **1**)	(**1**, **2**)	(**1**, **3**)	(**1**, **4**)	(**1**, **5**)	(**1**, **6**)
	2	(**2**, **1**)					
	3	(**3**, **1**)					
	4	(**4**, **1**)					
	5	(**5**, **1**)					
	6	(**6**, **1**)					

Maia's and Josh's variation: You can stop and accept your score at any time. Instead of playing all 12 turns, you may stop at any time and enjoy the score you have at the time you quit.

- Begin with 12 points. You roll double six on your first turn and now have a score of 24. You can stop and keep this score, or keep on rolling. What do you do?
- You begin with 12 points. You roll six non-doubles. Each turn you lost one point. Your score is 6. Do you stop and accept this score, or do you roll again and hope to get a double to add to your score?

A digit die (DD) is a 10-faced die with the faces numbered 0 to 9. Suppose you play Roll 2DD Doubles Game.

- Start with your score = 18.
- Roll 2DD each turn for 18 turns.
- Each turn: If the outcome is not a double, subtract 1 from your score. If the outcome is a double, add the sum of the double toyour score. Example: If you roll (7, 7), add 14 to your score.

Complete the table and calculate probabilities for the Roll 2DD Doubles Game. What is the probability of rolling a double?

		0	1	2	3	4	5	6	7	8	9
	0										
	1										
	2										
	3										
Green die	4										
	5										
	6										
	7										
	8										
	9										

Are All Perfect Numbers Triangular Numbers? | TOC

perfect number 1: a natural number n for which the sum of the factors of n is equal to $2n$.
2: a natural number n for which the sum of the proper factors of n is equal to n.

6 is a perfect number. The proper factors of 6 are 1, 2, and 3. 6 = 1 + 2 + 3

28 is a perfect number. The proper factors of 28 are 1, 2, 7, and 14. 28 = 1 + 2 + 3 + 14

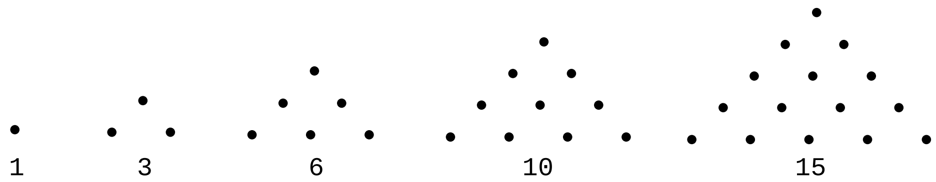
triangular number: the numbers 1, 3, 6, 10, 15, and so on. Triangular numbers can be represented by triangles having 1 dot, 3 dots, 6 dots, 10 dots, 15 dots, and so on. The first triangular number is 1. A triangular number greater than 1 is the sum of consecutive natural numbers beginning with 1.

3 is a triangular number. 3 = 1 + 2

6 is a triangular number. 6 = 1 + 2 + 3

10 is a triangular number. 10 = 1 + 2 + 3 + 4

15 is a triangular number. 15 = 1 + 2 + 3 + 4 + 5



Let T_n be the nth triangular number. $T_n = \frac{n(n+1)}{2}$

6 is the 1st perfect number	6 is the 3rd triangular number
6 = 1 + 2 + 3	$6 = \frac{3(3+1)}{2}$

28 is the 2nd perfect number	28 is the 7th triangular number
28 = 1 + 2 + 7 + 14	$28 = \frac{7(7+1)}{2}$

496 is the 3rd perfect number	496 is the 31st triangular number
496 = 1 = 2 + 4 + 8 + 16 + 31 + 62 + 124 + 248	$496 = \frac{31(31+1)}{2}$

The number 8128 is the 4th perfect number. Is 8128 a triangular number?

The number 33,550,336 is the 5th perfect number? Is 33,550,336 a triangular number?

Et cetera, et cetera. See a list of perfect numbers at
http://mathforum.org/library/drmath/view/51516.html

How to find out if a number is a perfect number

Use the triangular number formula: $T_n = \frac{n(n+1)}{2}$

Given a natural number *k*, is *k* a triangular number?

If yes, then $k = \frac{n(n+1)}{2}$ for some natural number *n*. Solve the equation for *n*.

1. $n(n+1) = 2k$ 2. $n^2 + n = 2k$ 3. $n^2 + n - 2k = 0$ A quadratic equation.

The standard form of the quadratic equation is $ax^2 + bx + c = 0$, where *a*, *b*, and *c* are real numbers, and $a \neq 0$. Use the **quadratic formula** to solve for *x*, given *a*, *b*, and *c*.

Quadratic formula in the variable *x*: $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

In our triangular number equation $n^2 + n - 2k = 0$, so *x* = *n*, *a* = 1, *b* = 1, and c = –2*k*:

Quadratic formula in the variable *n*: $n = \frac{-1 \pm \sqrt{1^2 - 4(1)(-2k)}}{2} = \frac{-1 \pm \sqrt{1^2 + 8k}}{2}$

Is 91 a triangular number? Set *k* = 91 and then plug and chug:

$$n = \frac{-1 \pm \sqrt{1^2 + 8(91)}}{2} = \frac{-1 \pm \sqrt{729}}{2} = \frac{-1 \pm 27}{2}$$

Solutions: *n* = 13 and *n* = –14. The solution *n* = –14 is not a natural number, so discard it.

The one-and-only solution is *n* = 13. Check: $\frac{13(13+1)}{2} = 91$ √ A-OK!

The number 91 is the 13th triangular number.

Is 76 a triangular number? Set k = 76 and then plug and chug:

$$n = \frac{-1 \pm \sqrt{1^2 + 8(76)}}{2} = \frac{-1 \pm \sqrt{609}}{2}$$

Solutions: n = 11.83896268 and n = –25.67792536.

These solutions are not natural numbers, so 76 is not a triangular number..

A shorter, quicker way to find out if a number is a triangular number

If k is a triangular number, then $k = \frac{n(n+1)}{2}$ for some natural number n, and $n^2 + n = 2k$.

If n is quite large, then n^2 is much greater than n and $n^2 \approx 2k$. [n^2 is approximately equal to $2k$.]

Good news! If $n^2 \approx 2k$, then $n \approx \sqrt{2k}$. That leads to the following algebraic alakazam to find out if k = 36 is a triangular number:

Algebraic Alakazam	**Example: *k* = 36**
1. Calculate the square root of 2k.	$\sqrt{2 \cdot 36} = \sqrt{72} \approx 8.48528$
2. Try the **integer part** of the square root as the value of n in the triangular number equation. Is $\frac{n(n+1)}{2} = k$?	$\frac{8(8+1)}{2} = \frac{8 \cdot 9}{2} = \frac{72}{2} = 36$ Huzzah! 36 is the 8th triangular number.

It worked. Now let's see if k = 123 is a triangular number.

Algebraic Alakazam	**Example: *k* = 123**
1. Calculate the square root of 2k.	$\sqrt{2 \cdot 123} = \sqrt{246} \approx 15.68439$
2. Try the **integer part** of the square root as the value of n in the triangular number equation. Is $\frac{n(n+1)}{2} = k$?	$\frac{15(15+1)}{2} = \frac{15 \cdot 16}{2} = \frac{240}{2} = 120 \neq 123$ 120 ≠ 123. 123 is not a triangular number.

Is perfect number 8128 a triangular number?

Algebraic Alakazam	**Example: *k* = 8128**
1. Calculate the square root of 2*k*. 2. Try the **integer part** of the square root as the value of *n* in the triangular number equation. Is $\frac{n(n+1)}{2} = k$?	$\sqrt{2 \cdot 8128} = \sqrt{16256} \approx 127.49902$ $\frac{127(127+1)}{2} = \frac{127 \cdot 128}{2} = \frac{16257}{2} = 8128$ Presto! 8128 is the 127th triangular number.

Is 100 a triangular number?

Algebraic Alakazam	**Example: *k* = 100**
1. Calculate the square root of 2*k*. 2. Try the **integer part** of the square root as the value of *n* in the triangular number equation. Is $\frac{n(n+1)}{2} = k$?	$\sqrt{2 \cdot 100} = \sqrt{200} \approx 14.14214$ $\frac{14(14+1)}{2} = \frac{14 \cdot 15}{2} = \frac{210}{2} = 105 \neq 100$ 100 is not a triangular number.

Hey! It even works for 3, the second triangular number, which is quite small.

Algebraic Alakazam	**Example: *k* = 3**
1. Calculate the square root of 2*k*. 2. Try the **integer part** of the square root as the value of *n* in the triangular number equation. Is $\frac{n(n+1)}{2} = k$?	$\sqrt{2 \cdot 3} = \sqrt{6} \approx 2.44949$ $\frac{2(2+1)}{2} = \frac{2 \cdot 3}{2} = \frac{6}{2} = 3$ Abracadabra! 3 is a triangular number.

Your Turn Which, if any, of these numbers are triangular numbers?

1. 1	**2**. 13	**3**. 1089
4. 1234	**5**. 2701	**6**. 4851
7. 20,100	**8**. 500,500	**9**. 33,550,336 [5th perfect no.]

A Word Game Using a TI-84 to 'Roll' 26-Faced Dice | TOC

Imagine 26-faced dice with the faces numbered 1 to 26. We didn't find any on the Internet. That's OK – we can use our TI-84 to 'roll' 26-faced dice.

Roll 1D26 (one 26-faced die). `MATH PRB randInt(1,26)Enter`

Roll 2D26 (two 26-faced dice). `MATH PRB randInt(1,26,2)Enter`

Roll 3D26 (three 26-faced die). `MATH PRB randInt(1,26,3)Enter`

Roll 4D26 (four 26-faced die). `MATH PRB randInt(1,26,4)Enter`

Roll 8D26 (eight 26-faced die). `MATH PRB randInt(1,26,8)Enter`

Assign a letter to each number from 1 to 26.

1 = A	2 = B	3 = C	4 = D	5 = E	6 = F	7 = G	8 = H	9 = I	10 = J
11 = K	12 = L	13 = M	14 = N	15 = O	16 = P	17 = Q	18 = R	19 = S	20 = T
21 = U	22 = V	23 = W	24 = X	25 = Y	26 = Z				

Roll 8D26 eight 26-faced dice. `randInt(1,26,8)Enter`

If you get fewer than 2 vowels, you may keep your roll or roll again. Vowels:

1	5	9	15	21
A	E	I	O	U

We rolled 8D26, converted the numbers to letters, and used the letters to make 1-letter, 2-letter, and 3-letter words.

Our best roll so far is 4, 9, 2, 5, 11, 1, 2, 5. We arranged them in ascending order and put the corresponding letter under each number.

```
1  2  2  4  5  5  9  11
A  B  B  D  E  E  I  K
```

We constructed words and checked them in the Scrabble dictionary.

1-letter words: A, I

2-letter words: AB, AD, BE, ED, ID

3-letter words: ADE, AID, BAD, BED, BEE, BID, DB, DEB, DIB, DIE, EBB, EEK, EKE, KID

Here are some alakazams about the words that we made.

ADE and AID are homonyms.

BED and DEB are permutations, anagrams, and semordnilaps.

BID and DIB are permutaions, anagrams, and semordnilaps.

EEK and EKE are permutations, anagrams, and homonyms.

BAD and DAB are permutations, anagrams, and semordnilaps.

KID is a child. Let us all work together to help a child create wonderment.

Your Turn Here are some 8D26 rolls. Make words.

19, 2, 15, 23, 8, 9, 20, 23

2, 14, 15, 1, 16, 18, 23, 16

12, 2, 21, 4, 7, 1, 22, 5

15, 18, 21, 16, 10, 25, 13, 1

13, 5, 12, 12, 22, 18, 10, 9

1, 3, 4, 26, 1, 10, 2, 5

22, 18, 5, 5, 25, 6, 21, 12

1 = A	2 = B	3 = C	4 = D	5 = E	6 = F	7 = G	8 = H	9 = I	10 = J
11 = K	12 = L	13 = M	14 = N	15 = O	16 = P	17 = Q	18 = R	19 = S	20 = T
21 = U	22 = V	23 = W	24 = X	25 = Y	26 = Z				

1	5	9	15	21
A	E	I	O	U

END | TOC