Explore/Investigate: Products of Proper Factors #1

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Reality expands to fill the available fantasies. To expand your personal reality, fantasize, fantasize, and keep on fantasizing. – Laran Stardrake

This **Mathemagical Alakazams** eBook is one of a bunch of eBooks that we are writing to share our wonderment about the magic of math with teachers, tutors, parents, siblings, anyone who helps a learner learn math. You can download this and other free math & science eBooks by Bob and George as PDF files or Microsoft Word files at **Information Age Education (IAE) Pedia**.

- http://i-a-e.org/downloads/cat_view/86-free-ebooks-by-bob-albrecht.html

At this IAE site, you can find Bob & George eBooks such as:

- *Mathemagical Meandering*
- *Mathemagical Black Holes*
- *Mathemagical Numbers 1 to 99*
- *Mathemagical Numbers 100 to 199*
- *Mathemagical Numbers 200 to 299*

Bob is an 84-year-old human (as of February 2014). George is a Dragon. Read about Bob & George at **Information Age Education** (**IAE**): http://iae-pedia.org/Robert_Albrecht

Bob was once a kid in Greene, Iowa, home to 1300 denizens. He day-dreamed about exploring and adventuring. Bob planned in great detail imaginary adventures such as canoeing from the west coast of the USA to the east coast of the USA. Six adventurers in three canoes. If they lost a canoe, they could continue with three adventurers in each remaining canoe until they reached a place where they could acquire another canoe.

Bob is now a bit old to go adventuring in a canoe into the wilderness. That is AOK because adventure beckons in mathemagical land, a wonderful wilderness for kids of all ages.

Information Age Education (http://iae-pedia.org/Main_Page) publishes a large number of free books, a free blog, the free IAE-Pedia, and the free IAE Newsletter.

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Table of Contents (TOC)

Play Together, Learn Together

DragonFun image by Marcie Hawthorne http://marciehawthorne.com/

Products of Proper Factors Internet Quest – No Joy | TOC

We have known a bit about **sums** of proper factors of natural numbers for a long time. We assume that you do too, but in case you don't understand everything you know about factors, proper factors, sums of proper factors, deficient numbers, abundant numbers, and perfect numbers, we will contrive a brief review down yonder.

We frequently awake during the night wondering about mathemagical alakazams. Sometimes we leap (well, slowly ooze) out of bed and jot down a bunch of notes to remind us to check out an idea. One night we woke in the wee hours wondering about **products** of proper factors of natural numbers. We eased out of bed and wrote a note to look it up on the Internet. Later that day, we cranked up our computer and searched for

- product of proper factors of a natural number

Oops. We found lots of stuff about factors and proper factors and **sums** of proper factors, but found zilch, nada, nothing about **products** of proper factors of numbers. We did another search:

- sum of proper factors of a natural number

Lots of hits – 12 million hits!

As we think you know, the sum of the proper factors of a natural number greater than 1 classifies that number as a **deficient number**, **abundant number**, or **perfect number**. We will review these numbers down yonder. Or – click your way to knowhow on the Internet.

- Deficient Number http://en.wikipedia.org/wiki/Deficient_number
- Abundant Number http://en.wikipedia.org/wiki/Abundant_number
- Perfect Number http://en.wikipedia.org/wiki/Perfect_number

On the Internet, there is much ado about **sums** of proper factors of natural numbers, but we did not find anything about **products** of proper factors of natural numbers.

⚐⚐⚐	Aha! Perhaps this is unexplored territory. If true, let's explore together. In this eBook, we will begin to explore this unexplored (?) territory and share our exploration with you. We will leave much exploring undone and hope that **YOU** and **YOUR STUDENTS** will continue the exploration. Is this exploration an investigation? We think of your students as explorers. We also think of them as investigators. So frequently in this eBook, we will call them • explorers/investigators	⚐⚐⚐

Review: Sums of Proper Factors of Natural Numbers | TOC

⚐⚐⚐	Hey! If you already know about factors, proper factors, sums of proper factors, deficient numbers, abundant numbers, and perfect numbers, scroll on down to **Products of Proper Factors of Natural Numbers**.	⚐⚐⚐

factor If you multiply two natural numbers, the product is a natural number. The numbers you multiplied to obtain the product are factors of the product. Examples: 2 x 3 = 6, so 2 and 3 are factors of 6. 1 x 6 = 6, so 1 and 6 are factors of 6. The factors of 6 are 1, 2, 3, and 6.

- If $a \times b = c$, then a and b are factors of c.

proper factor a factor of a natural number other than the number itself. A proper factor of a number is a factor that is less than the number. The proper factors of 6 are 1, 2, and 3.

deficient number a natural number n for which the sum of the proper factors of n is less than n. Example: The proper factors of 10 are 1, 2, and 5. 1 + 2 + 5 = 8 and 8 is less than 10. 10 is a deficient number.

abundant number a natural number n for which the sum of the proper factors of n is greater than n. Example: The proper factors of 12 are 1, 2, 3, 4, and 6. 1 + 2 + 3 + 4 + 6 = 16 and 16 is greater than 12. 12 is an abundant number.

perfect number a natural number n for which the sum of the proper factors of n is equal to n. Example: The proper factors of 6 are 1, 2, and 3. 1 + 2 + 3 = 6. 6 is a perfect number.

no type number a natural number that is not a deficient number, abundant number, or perfect number. The one-and-only 'no type' number is 1. 1 has no proper factors.

A deficient number is not a number with a personality defect. Deficient numbers can be very mathemagical. For example, ALL prime numbers are deficient numbers. The one-and-only proper factor of a prime number is 1, so the sum of proper factors of a prime number is exactly 1. For every prime number n, the sum of proper factors of n is 1 and 1 is less than n.

What do you think about *abundant* as the tag for a number that has a sum of proper factors greater than the number? Look up *abundant* in a dictionary. Can you think of a better name?

Perfect numbers are rare. The first (least) four perfect numbers are 6, 28, 496, and 8128. When we lived in Oregon, by a lucky whim of fate, Bob's car license plate ended in 496. Before moving on, you might enjoy browsing these Internet sites that we mentioned earlier.

- Deficient Number http://en.wikipedia.org/wiki/Deficient_number
- Abundant Number http://en.wikipedia.org/wiki/Abundant_number
- Perfect Number http://en.wikipedia.org/wiki/Perfect_number

We love tables, so we will contrive a table of natural numbers (*N*), proper factors of *N*, sum of proper factors of *N*, and *N*'s type of number (deficient, abundant, perfect, or no type).

Table 01 *N*, Proper Factors of *N*, Sum of Proper Factors of *N*, and *N*'s Type of Number			
N	**Proper Factors of *N***	**Sum of Proper Factors of *N***	***N*'s Type of Number**
1	none	none	no type
2	1	1 [See note below table.]	deficient (1 < 2)
3	1	1	deficient (1 < 3)
4	1, 2	1 + 2 = 3	deficient (3 < 4)
5	1	1	deficient (1 < 5)
6	1, 2, 3	1 + 2 + 4 = 6	perfect (6 = 6)
7	1	1	deficient (1 < 7)
8	1, 2, 4	1 + 2 + 4 = 7	deficient (7 < 8)
9	1, 3	1 + 3 = 4	deficient (4 < 9)
10	1, 2, 5	1 + 2 + 5 = 8	deficient (8 < 10)
11	1	1	deficient (1 < 11)
12	1, 2, 3, 4, 6	1 + 2 + 3 + 4 + 6 = 16	abundant (16 > 12)
13	1	1	deficient (1 < 13)
14	1, 2, 7	1 + 2 + 7 = 10	deficient (10 < 14)
15	1, 3, 5	1 + 3 + 5 = 9	deficient (9 < 15)
16	1, 2, 4, 8	1 + 2 + 4 + 8 = 15	deficient (15 < 16)
17	1	1	deficient (1 < 17)
18	1, 2, 3, 6, 9	1 + 2 + 3 + 6 + 9 = 21	abundant (21 > 18)
19	1	1	deficient (1 < 19)
20	1, 2, 4, 5, 10	1 + 2 + 4 + 5 + 10 = 22	abundant (22 > 20)
21	1, 3, 7	1 + 3 + 7 = 11	deficient (11 < 21)
22	1, 2, 11	1 + 2 + 11 = 14	deficient (14 < 22)
23	1	1	deficient (1 < 23)
24	1, 2, 3, 4, 6, 8, 12	1 + 2 + 3 + 4 + 6 + 8 + 12 = 36	abundant (36 > 24)
25	1, 5	1 + 5 = 6	deficient (6 < 25)

Note. Is a sum the sum of two or more numbers? After a bit of perplexed pondering, we decided to define the sum of one number as that number. The only proper factor of a prime number is 1, so we define the sum of proper factors of a prime number as 1. Okay? Please say yes.

Your Turn 01. Browse the table of natural numbers 1 to 25 (Table 01) and do these exercises. Our answers are on the next page. **Don't peek!**

1. If the sum of the proper factors of a natural number *N* is less than *N*,

   then *N* is a/an _______________ number.

2. If the sum of the proper factors of a natural number *N* is greater than *N*,

   then *N* is a/an _______________ number.

3. If the sum of the proper factors of a natural number *N* is equal to *N*,

   then *N* is a/an _______________ number.

4. In the table of numbers 1 to 25 (Table 01), how many numbers are deficient numbers? ____

5. In the table of numbers 1 to 25 (Table 01), how many numbers are abundant numbers? ____

6. In the table of numbers 1 to 25 (Table 01), how many numbers are perfect numbers? ____

7. In the table of numbers 1 to 25 (Table 01), how many numbers are 'no type' numbers? ____

8. Complete the following **frequency table** that shows the number of deficient numbers, the number of abundant numbers, the number of perfect numbers, the number of 'no type' numbers, and the total number of numbers in the table of natural numbers 1 to 25 (Table 01).

   - Frequency Distribution http://en.wikipedia.org/wiki/Frequency_distribution

Type of number	deficient	abundant	perfect	no type	total
Frequency					25

9. Use the data in the frequency table in 8 (above) to calculate the **experimental probability of occurrence** of deficient numbers, abundant numbers, perfect numbers, and 'no type' numbers in the set of natural numbers 1 to 25 (Table 01).

   - Experimental Probability Wikipedia calls this **empirical probability**. http://en.wikipedia.org/wiki/Empirical_probability
   - Experimental probability of occurrence of a deficient number = ___________
   - Experimental probability of occurrence of an abundant number = ___________
   - Experimental probability of occurrence of a perfect number = ___________
   - Experimental probability of occurrence of a 'no type' number = ___________

Answers 01.

1. If the sum of the proper factors of a natural number *N* is less than *N*, then *N* is a/an deficient number.

2. If the sum of the proper factors of a natural number *N* is greater than *N*, then *N* is a/an abundant number.

3. If the sum of the proper factors of a natural number *N* is equal to *N*, then *N* is a/an perfect number.

4. In the table of numbers 1 to 25 (Table 01), how many numbers are deficient numbers? 19

5. In the table of numbers 1 to 25 (Table 01), how many numbers are abundant numbers? 4

6. In the table of numbers 1 to 25 (Table 01), how many numbers are perfect numbers? 1

7. In the table of numbers 1 to 25 (Table 01), how many numbers are 'no type' numbers? 1

8. Complete the following **frequency table** that shows the number of deficient numbers, the number of abundant numbers, the number of perfect numbers, the number of 'no type' numbers, and the total number of numbers in the table of natural numbers 1 to 25 (Table 01).

Type of number	deficient	abundant	perfect	no type	total
Frequency	19	4	1	1	25

9. Use the data in the frequency table in 8 (above) to calculate the **experimental probability of occurrence** of deficient numbers, abundant numbers, perfect numbers, and 'no type' numbers in the set of natural numbers 1 to 25 (Table 01).

- Experimental probability of occurrence of a deficient number = 19/25
- Experimental probability of occurrence of an abundant number = 4/25
- Experimental probability of occurrence of a perfect number = 1/25
- Experimental probability of occurrence of a 'no type' number = 1/25

Ahoy Teacher, We suggest that you discuss **experimental probability** and **theoretical probability** with your students.

- Theoretical versus experimental probability

  http://www.algebra-class.com/theoretical-probability.html

Are these experimental probabilities good predictors of the numbers of deficient numbers, abundant numbers, perfect numbers, and 'no type' numbers in a different set of natural numbers, such as 1 to 100 or 1 to 1000 or 123 to 789? That's another eBook for another time.

Your Turn 02 Table 02 lists natural numbers 1 to 50, their proper factors, the sums of their proper factors, and an empty column labeled **Type** in which **YOU** can write

- D for deficient number, A for abundant number, P for perfect number or N for 'no type' number.

Abbreviations *N*: natural number, PF: proper factor(s), Sum PF: sum of proper factors.

Table 02 Numbers 1 to 50, Proper Factors, Sum of Proper Factors, and Type of Number

N	PF of *N*	Sum PF	Type		*N*	PF of *N*	Sum PF	Type
1	none	none			26	1, 2, 13	16	
2	1	1			27	1, 3, 9	13	
3	1	1			28	1, 2, 4, 7, 14	28	
4	1, 2	3			29	1	1	
5	1	1			30	1, 2, 3, 5, 6, 10, 15	42	
6	1, 2, 3	6			31	1	1	
7	7	1			32	1, 2, 4, 8, 16	31	
8	1, 2, 4	7			33	1, 3, 11	15	
9	1, 3	4			34	1, 2, 17	20	
10	1, 2, 5	8			35	1, 5, 7	13	
11	1	1			36	1, 2, 3, 4, 6, 9, 12, 18	55	
12	1, 2, 3, 4, 6	16			37	1	1	
13	1	1			38	1, 2, 19	22	
14	1, 2, 7	10			39	1, 3, 13	17	
15	1, 3, 5	9			40	1, 2, 4, 5, 8, 10, 20	50	
16	1, 2, 4, 8	15			41	1	1	
17	1	1			42	1, 2, 3, 6, 7, 14, 21	54	
18	1, 2, 3, 6, 9	21			43	1	1	
19	1	1			44	1, 2, 4, 11, 22	40	
20	1, 2, 4, 5, 10	22			45	1, 3, 5, 9, 15	33	
21	1, 3, 7	11			46	1, 2, 23	26	
22	1, 2, 11	14			47	1	1	
23	1	1			48	1, 2, 3, 4, 6, 8, 12, 16, 24	76	
24	1, 2, 3, 4, 6, 8, 12	36			49	1, 7	8	
25	1, 5	6			50	1, 2, 5, 10, 25	43	

Answers 02 Abbreviations: *N*: number, PF: proper factor(s), Sum PF: sum of proper factors.

Table 02 Numbers 1 to 50, Proper Factors, Sum of Proper Factors, and Type of Number								
N	**PF of *N***	**Sum PF**	**Type**		***N***	**PF of *N***	**Sum PF**	**Type**
1	none	none	N		26	1, 2, 13	16	D
2	1	1	D		27	1, 3, 9	13	D
3	1	1	D		28	1, 2, 4, 7, 14	28	P
4	1, 2	3	D		29	1	1	D
5	1	1	D		30	1, 2, 3, 5, 6, 10, 15	42	A
6	1, 2, 3	6	P		31	1	1	D
7	7	1	D		32	1, 2, 4, 8, 16	31	D
8	1, 2, 4	7	D		33	1, 3, 11	15	D
9	1, 3	4	D		34	1, 2, 17	20	D
10	1, 2, 5	8	D		35	1, 5, 7	13	D
11	1	1	D		36	1, 2, 3, 4, 6, 9, 12, 18	55	A
12	1, 2, 3, 4, 6	16	A		37	1	1	D
13	1	1	D		38	1, 2, 19	22	D
14	1, 2, 7	10	D		39	1, 3, 13	17	D
15	1, 3, 5	9	D		40	1, 2, 4, 5, 8, 10, 20	50	A
16	1, 2, 4, 8	15	D		41	1	1	D
17	1	1	D		42	1, 2, 3, 6, 7, 14, 21	54	A
18	1, 2, 3, 6, 9	21	A		43	1	1	D
19	1	1	D		44	1, 2, 4, 11, 22	40	D
20	1, 2, 4, 5, 10	22	A		45	1, 3, 5, 9, 15	33	D
21	1, 3, 7	11	D		46	1, 2, 23	26	D
22	1, 2, 11	14	D		47	1	1	D
23	1	1	D		48	1, 2, 3, 4, 6, 8, 12, 16, 24	76	A
24	1, 2, 3, 4, 6, 8, 12	36	A		49	1, 7	8	D
25	1, 5	6	D		50	1, 2, 5, 10, 25	43	D

End of sum of proper products of natural numbers review.
Onward, intrepid explorers, adventure into **new territory:**
Products of proper factors of natural numbers.

Products of Proper Factors of Natural Numbers | TOC

Earlier we said, and now repeat: We frequently awake during the night wondering about mathemagical alakazams. Sometimes we leap (well, ooze) out of bed and jot down a bunch of notes to remind us to check out an idea. One night we woke in the wee hours wondering about **products** of proper factors of natural numbers. We eased out of bed and wrote a note to look it up on the Internet. Later that day, we cranked up our computer and searched for

- product of proper factors of natural numbers

Oops. We found lots of stuff about factors and proper factors and **sums** of proper factors, but found zilch, nada, nothing about **products** of proper factors of natural numbers. On the Internet, there is much ado about **sums** of proper factors of natural numbers, but we did not find anything about **products** of proper factors of natural numbers. [Summer 2014]

⚐⚐⚐	Aha! Perhaps this is unexplored territory. If true, let's explore together. In this eBook, we will begin exploring this unexplored (?) territory and share our exploration with you. We will leave much exploring undone and hope that **YOU** and **YOUR STUDENTS** will continue the exploration. We think of your students as explorers. We also think of them as investigators. So frequently in this eBook, we will call them • explorers/investigators	⚐⚐⚐

At ages 15 and 19, Bob explored the Boundary Waters Canoe Area Wilderness in canoes, first as a Boy Scout in a group of Boy Scouts and later with good friend SG. Together Bob and SG paddled and portaged 100 miles in 5 days from Ely, MN northward through many lakes, and back to Ely. An adventure that Bob will treasure forever.

Bob plays back long ago adventures in his mind and imagines canoeing into mathemagical land and exploring products of proper factors.

Picture: Wikipedia http://en.wikipedia.org/wiki/Boundary_Waters_Canoe_Area_Wilderness.

Go to Bing or Google, search for **Boundary Waters Canoe Area Wilderness**. Click on **Images** to see many pictures of the Boundary Waters Canoe Area Wilderness. Explore the Boundary Waters in your imaginary canoe and then move on and explore **products of proper factors**.

Table 03 shows natural numbers 1 to 25 (*N*), proper factors of *N* (PF of *N*), number of proper factors of *N* (# PF), product of proper factors of *N* (Product of PF), and product of proper factors written as a power of *N* (N^{power}). We use our TI-84 to crunch numbers, so in Table 03 we use the TI-84's multiplication symbol (*) to indicate multiplication Example: 1 * 2 * 3 = 6.

Table 03 *N*, Proper Factors, Number and Product of Proper Factors of *N*, *N* to a Power

N	PF of *N*	# PF	Product of PF	N^{power}
1	none	none	none	---
2	1	1	1	2^0
3	1	1	1	3^0
4	1, 2	2	1 * 2 = 2	$4^{1/2}$
5	1	1	1	5^0
6	1, 2, 3	3	1 * 2 * 3 = 6	6^1
7	1	1	1	7^0
8	1, 2, 4	3	1 * 2 * 4 = 8	8^1
9	1, 3	2	1 * 3 = 3	$9^{1/2}$
10	1, 2, 5	3	1 * 2 * 5 = 10	10^1
11	1	1	1	11^0
12	1, 2, 3, 4, 6	5	1 * 2 * 3 * 4 * 6 = 144	12^2
13	1	1	1	13^0
14	1, 2, 7	3	1 * 2 * 7 = 14	14^1
15	1, 3, 5	3	1 * 3 * 5 = 15	15^1
16	1, 2, 4, 8	4	1 * 2 * 4 * 8 = 64	8^2
17	1	1	1	17^0
18	1, 2, 3, 6, 9	5	1 * 2 * 3 * 6 * 9 = 324	18^2
19	1	1	1	19^0
20	1, 2, 4, 5, 10	5	1 * 2 * 4 * 5 * 10 = 400	20^2
21	1, 3, 7	3	1 * 3 * 7 = 21	21^1
22	1, 2, 11	3	1 * 2 * 11 = 22	22^1
23	1	1	1	19^0
24	1, 2, 3, 4, 6, 8, 12	7	1 * 2 * 3 * 4 * 6 * 8 * 12 = 13,824	24^3
25	1, 5	2	1 * 5 = 5	$25^{1/2}$

Note. Is a product usually the product of two or more numbers? We define the product of one number as that number. The only proper factor of a prime number is 1, so we define the product of proper factors of a prime number as 1. Okay? Please say yes.

As we write this eBook (summer 2014), we could not find information about products of proper factors on the Internet. Amazing – we wonder if and where someone has explored products of proper factors of natural numbers.

Your explorers/investigators (aka students) can explore products of proper factors, post their work on the Internet, submit articles to the school newspaper, math journals such as the journal of your state NCTM affiliate (See http://www.nctm.org/), et cetera, et cetera.

Math is a game of patterns. Your explorers can explore Table 03, make **observations**, and discover patterns. Look in the N^{power} column to find data that might elicit an Aha! or Eureka! AFTER they peruse Table 03 and venture forth **their** observations and conjectures, ask them to peruse the data again, make more observations and conjectures, and then compare their findings with those of other investigators. Observations? Conjectures? See

- Observation http://en.wikipedia.org/wiki/Observation
- Conjecture http://en.wikipedia.org/wiki/Conjecture

A conjecture can be true or false. A conjecture based on the data in Table 03 might be true for the set of natural numbers 1 to 25, but false for a different set, such as natural numbers 1 to 100 or 100 to 200, or *m* to *n*, where your investigators choose the values of *m* and *n*.

If your explorers/investigators are a bit boggled about finding patterns, you might s l o w l y point to trailheads into the wilderness, trailheads that **you create** or trailheads that we suggest here and on the next bunch of pages.

In Table 03, there are prime numbers 2, 3, 5, 7, 11, 13, 17, 19, and 23, wonderful critters in the world of mathemagical numbers. For each and every one of these prime numbers:

- Observation: In Table 03, the one-and-only proper factor of a prime number is 1.
- Observation: In Table 03, the number of proper factors of a prime number is 1.
- Observation: In Table 03, the product of proper factors of a prime number is 1.
- Conjecture: If N is a prime number, then N has 1 proper factor.
- Conjecture: If N is a prime number, then the product of proper factors of $N = 1$.
- Conjecture: If N is a prime number, then the product of proper factors of $N = N^0$.

Ahoy Teacher, The conjectures are true for all prime numbers. Why? A prime number N has exactly 1 proper factor. The proper factor is 1. The product of proper factors of a prime number is $N^0 = 1$.

Table 04 presents more data that you and your students can use to discover patterns.

Table 04 *N*, Proper Factors, Number and Product of Proper Factors of *N*, *N* to a Power

N	PF of *N*	# PF	Product of PF	N^{power}
26	1, 2, 13	3	1 * 2 * 13 = 26	26^{1}
27	1, 3, 9	3	1 * 3 * 9 = 27	27^{1}
28	1, 2, 4, 7, 14	5	1 * 2 * 4 * 7 * 14 = 784	28^{2}
29	1	1	1	29^{0}
30	1, 2, 3, 5, 6, 10, 15	7	1 * 2 * 3 * 5 * 6 * 10 * 15 = 27,000	30^{3}
31	1	1	1	31^{0}
32	1, 2, 4, 8, 16	5	1 * 2 * 4 * 8 * 16 = 1024	32^{2}
33	1, 3, 11	3	1 * 3 * 11 = 33	33^{1}
34	1, 2, 17	3	1 * 2 * 17 = 34	34^{1}
35	1, 5, 7	3	1 * 5 * 7 = 35	35^{1}
36	1, 2, 3, 4, 6, 9, 12, 18	8	1 * 2 * 3 * 4 * 6 * 9 * 12 * 18 = 279,936	$36^{7/2}$
37	1	1	1	37^{0}
38	1, 2, 19	3	1 * 2 * 19 = 38	38^{1}
39	1, 3, 13	3	1 * 3 * 13 = 39	39^{1}
40	1, 2, 4, 5, 8, 10, 20	7	1 * 2 * 4 * 5 * 8 * 10 * 20 = 64,000	40^{3}
41	1	1	1	41^{0}
42	1, 2, 3, 6, 7, 14, 21	4	1 * 2* 3 * 6 * 7 * 14 * 21 =70,488	42^{3}
43	1	1	1	43^{0}
44	1, 2, 4, 11, 22	5	1 * 2 * 4 * 11 * 22 = 1936	44^{2}
45	1, 3, 5, 9, 15	5	1 * 3 * 5 * 9 * 15 = 2025	45^{2}
46	1, 2, 23	3	1 * 2 * 23 = 46	46^{1}
47	1	1	1	47^{0}
48	1, 2, 3, 4, 6, 8, 12, 16, 24	9	1 * 2 * 3 * 4 * 6 * 8 * 12 * 16 * 24 = 5,308,416	48^{4}
49	1, 7	2	1 * 7 = 7	$49^{1/2}$
50	1, 2, 5, 10, 25	5	1 * 2 * 5 * 10 * 25 = 2500	50^{2}

Your explorers/investigators can explore Tables 03 and 04, make observations, find patterns, and make conjectures. Hey! Interesting observations are great (give credit) and easier to make than conjectures (give more credit). Here is a far-out observation that we hope that some of your investigators glom onto:

- The proper factors of 6 are 1, 2, and 3. The sum of proper factors of 6 is 1 + 2 + 3 = 6 and the product of proper factors of 6 is 1 * 2 * 3 = 6. Is 6 the only natural number for which this is true? Is there another natural for which the sum of its proper factors is equal to the product of its proper factors? We don't know. Investigate!

A natural number might have one proper factor, two proper factors, three proper factors, four proper factors, or more proper factors. Students who are having difficulty finding patterns might enjoy browsing Table 05 of numbers we copied from Tables 03 and 04.

Table 05 Natural Numbers That have Exactly One Proper Factor (Prime Numbers)

N	**PF of *N***	**# PF**	**Product of PF**	***N* to a power**
2	1	1	1	2^0
3	1	1	1	3^0
5	1	1	1	5^0
7	1	1	1	7^0
11	1	1	1	11^0
13	1	1	1	13^0
17	1	1	1	17^0
19	1	1	1	19^0
23	1	1	1	23^0
29	1	1	1	29^0
31	1	1	1	31^0
37	1	1	1	37^0

We hope your explorers/investigators make observations and conjectures, perhaps with a gentle nudge from you. We made these observations and conjectures about numbers in Table 05.

- Observation: Every prime number in Table 05 has exactly one proper factor.
- Observation: The one-and-only proper factor of every prime number in Table 05 is 1.
- Observation: The product of proper factors of every prime number in Table 05 is $1 = N^0$.
- Conjecture: If N is a prime number, then the number of proper factors of N is 1.
- Conjecture: If N is a prime number, then the product of proper factors of N is $N^0 = 1$.

It will be great if your students explore these conjectures for prime numbers that are not shown in Table 05, such as 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97, et cetera, et cetera.

prime number **1**. a natural number that has exactly two different factors. **2**. a natural number that has exactly one proper factor.

What about numbers that have exactly two proper factors? We copied numbers that have two proper factors from Tables 03 and 04 and stuffed them into Table 06.

Table 06 Natural Numbers That Have Exactly Two Proper Factors

N	**PFof *N***	**# PF**	**Product of PF**	***N* power**
4	1, 2	2	1 * 2 = 2	$4^{1/2}$
9	1, 3	2	1 * 3 = 3	$9^{1/2}$
25	1, 5	2	1 * 5 = 5	$4^{1/2}$
49	1, 7	2	1 * 7 = 7	$49^{1/2}$

We hope your explorers/investigators observe, perhaps with a bit of guidance from you:

- Observation: Every number *N* in Table 06 has exactly two proper factors.
- Observation: One of the proper factors is a prime number.
- Observation: Every number *N* in Table 06 is a composite number.
- Observation: 4, 9, 25, and 49 are the squares of the prime numbers 2, 3, 5, and 7.

- Conjecture: The square of a prime number has exactly two proper factors.
- Conjecture: If a natural number *N* has exactly two proper factors, then the product of the proper factors of *N* is equal to the square root of *N*.
- Conjecture: If a natural number *N* is the square of a prime number, then the product of the proper factors of *N* is equal to $N^{1/2}$.

The square numbers 16 and 36 each have more than two proper factors.

16	1, 2, 4, 8	4	1 * 2 * 4 * 8 = 64	8^{2}
36	1, 2, 3, 4, 6, 9, 12, 18	8	1 * 2 * 3 * 4 * 6 * 9 * 12 * 18 = 279,936	$36^{7/2}$

4, 9, 16, 25, 36, and 49 are **square numbers**. Why do square numbers 4, 9, 25, and 49 each have two proper factors, but square numbers 16 and 36 have more than two proper factors?

It will be scrumptious if your explorers/investigations observe that each square number 4, 9, 16, 25, and 36 has an **even number** of proper factors and make a conjecture such as:

- Conjecture: If *N* is a square number, then *N* has an even number of proper factors.

Your explorers/investigators can browse Tables 03 & 04, and construct tables of numbers that have exactly three proper factors, four proper factors, or more proper factors. We will too. It will be cool if they create their tables **before** they see our tables. Here is an "empty" table that you can copy and hand out. Remember: You may edit this eBook, and give snippets or the entire eBook free to anyone. It is AOK to paste this table into **your document**, edit it, print it, and hand it out to your investigators.

Table ___ Natural Numbers That Have Exactly ________ Proper Factors				
N	**PF of *N***	**# PF**	**Product of PF**	***N*** $^{\text{power}}$

Whoa! We hope that your investigators create **their tables** of numbers that have exactly three proper factors, four proper factors, or more proper factors **before** they see our tables on this page and the next bunch of pages.

Table 07 Natural Numbers That Have Exactly Three Proper Factors				
N	**PF of *N***	**# PF**	**Product of PF**	***N*** ^{power}
6	1, 2, 3	3	1 * 2 * 3 = 6	6^1
8	1, 2, 4	3	1 * 2 * 4 = 8	8^1
10	1, 2, 5	3	1 * 2 * 5 = 10	10^1
14	1, 2, 7	3	1 * 2 * 7 = 14	14^1
15	1, 3, 5	3	1 * 3 * 5 = 15	15^1
21	1, 3, 7	3	1 * 3 * 7 = 21	21^1
22	1, 2, 11	3	1 * 2 * 11 = 22	22^1
26	1, 2, 13	3	1 * 2 * 13 = 26	26^1
27	1, 3, 9	3	1 * 3 * 9 = 27	27^1
33	1, 3, 11	3	1 * 3 * 11 = 33	33^1
34	1, 2, 17	3	1 * 2 * 17 = 34	34^1
35	1, 5, 7	3	1 * 5 * 7 = 35	35^1
38	1, 2, 19	3	1 * 2 * 19 = 38	38^1
39	1, 3, 13	3	1 * 3 * 13 = 39	39^1

Three is a popular number of proper factors. If your explorers carefully cruise Table 07, what patterns might they find? We suggest (Beware! conjectures may be true or false):

- Observation: Every number *N* in Table 07 is a composite number.
- Observation: Every number *N* in Table 07 has exactly three proper factors.
- Observation: For every number *N* in Table 07, the product of proper factors of $N = N^1$.
- Observation: For every number *N* in Table 07, two of the three proper factors are prime numbers.
- Conjecture: If a natural number *N* has exactly three proper factors, then the product of the proper factors of *N* is equal to $N = N^1$.
- Conjecture: If the product of the proper factors of a natural number *N* is equal to *N*, Then *N* has exactly three proper factors.
- Conjecture: If a natural number *N* has exactly three proper factors, then two of the proper factors are prime numbers.

Ahoy investigators. If a number *N* has exactly three proper factors, then one of the proper factors must be 1. If the other two proper factors are two **different** prime numbers, is the product of proper factors equal to *N*? Investigate:

The first 25 prime numbers are

- 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71,73, 79, 83, 89, 97

Construct numbers that have exactly three proper factors, where the proper factors are 1 and two different prime numbers. Here are a few:

1 * 2 * 97 = 194	3 proper factors. 2 proper factors are prime numbers.
1 * 3 * 73 = 219	3 proper factors. 2 proper factors are prime numbers.
1 * 5 * 71 = 355	3 proper factors. 2 proper factors are prime numbers.
1 * 7 * 83 = 581	3 proper factors. 2 proper factors are prime numbers.
1 * 89 * 97 = 8633	3 proper factors. 2 proper factors are prime numbers.

How many numbers can you construct that have three proper factors consisting of 1 and two different prime numbers selected from the first 25 prime numbers? How many ways are there to select two different thingamajigs from a set of 25 different thingamajigs?

Similar problem: How many ways are there to select two different letters from the 26 letters of the alphabet?

Yup, you guessed it. These are **combination** problems.

- See Combination http://en.wikipedia.org/wiki/Combination

⚐⚐⚐	Ahoy Teacher, We write this eBook for **YOU**. We provide many observations, suggestions, and conjectures as data that, we hope, you can use to interest your students in exploring **products of proper factors**. We could not find information about products of proper factors on the Internet, so it would be great if your investigators explore this territory and post their results on the Internet. Hey! What say! Products of proper factors on **Facebook**? Can your explorers/investigators boggle a bunch of Facebook browsers? You, as explorer/investigator guide/coach can organize teams, equip them for exploration, and point each team toward a path into the unknown. For us, this is another adventure in a lifetime of adventures.	⚐⚐⚐

Tables 03 and 04 list numbers 1 to 50. Alas, in Tables 03 and 04, we found only one natural number that has exactly four proper factors:

16	1, 2, 4, 8	4	1 * 2 * 4 * 8 = 64	8^{2}

We investigated natural numbers greater than 50 by browsing Appendix 01 Proper Factors of Numbers 1 to 300. We found one more natural number (81) that has exactly four proper factors. Then – aha! – we thought we saw a pattern. Using the pattern we thought we saw, we found more natural numbers that have exactly four proper factors. Here are some of them in Table 08.

Table 08 Natural Numbers That Have Exactly Four Proper Factors

N	**PF of *N***	**# PF**	**Product of PF**	N^{power}
16	1, 2, 4, 8	4	1 * 2 * 4 * 8 = 64	$16^{3/2}$
81	1, 3, 9, 27	4	1 3 * 9 * 27 = 729	$81^{3/2}$
625	1, 5, 25, 125	4	1 * 5 * 25 * 125 = 15,625	$5^{3/2}$
2401	1, 7, 49, 343	4	1 * 7 * 49 * 343 = 117,649	$7^{3/2}$

What do your investigators observe when they examine the numbers in Table 08? We suggest:

- Observation: Every natural number in Table 08 has exactly four proper factors.
- Observation: Every number in Table 08 is the fourth power of a prime number.

  $16 = 2^{4}$ $81 = 3^{4}$ $625 = 5^{4}$ $2401 = 7^{4}$

- Conjecture: If a natural number N has exactly four proper factors, then N is equal to the 4th power of a prime number. $N = (\text{prime number})^{4}$
- Conjecture: If a natural number N is equal to the 4th power of a prime number, then N has exactly four proper factors.
- Conjecture: If a natural number N has exactly four proper factors, then the product of the proper factors is equal to $N^{3/2}$. [Product of proper factors of $N = N^{3/2}$]

To conjecture and be wrong is better than never to have conjectured.
To show that a conjecture is false, simply find one counterexample.

- Counterexample http://en.wikipedia.org/wiki/Counterexample

The numbers 16, 81, 625, and 2401 are the 4th powers of the first four prime numbers 2, 3, 5, and 7. What say you and your explorers investigate the above conjectures for the next four prime numbers: 11, 13, 17, and 19? Products of their proper factors are very large numbers!

On to numbers in Tables 03 and 04 that have exactly five proper factors. Here we go:

Table 09 Natural Numbers That Have Exactly Five Proper Factors				
N	**PF of *N***	**# PF**	**Product of PF**	N^{power}
12	1, 2, 3, 4, 6	5	1 * 2 * 3 * 4 * 6 = 144	12^2
18	1, 2, 3, 6, 9	5	1 * 2 * 3 * 6 * 9 = 324	18^2
20	1, 2, 4, 5, 10	5	1 * 2 * 4 * 5 * 10 = 400	20^2
28	1, 2, 4, 7, 14	5	1 * 2 * 4 * 7 * 14 = 784	28^2
32	1, 2, 4, 8, 16	5	1 * 2 * 4 * 8 * 16 = 1024	32^2

Observations and conjectures (conjectures may be true or false):

- Observation: Every natural number in Table 09 has exactly five proper factors.
- Observation: Every natural number in Table 09 is a composite number.
- Observation: No number in Table 09 is a square number.
- Conjecture: If a natural number N has exactly five proper factors, then the product of proper factors of N is equal to the square of N. [Product of proper factors of $N = N^2$].
- Conjecture: If the product of the proper factors of a natural number N is equal to N^2, then N has exactly five proper factors.
- Far-out conjecture: If a natural number N is a composite number and is not a square number, then N has an odd number of proper factors.

We suggest that your explorers/investigators make tables that summarize information in previous tables, especially the products of proper factors of a natural number N as a power of N. AFTER they create their tables, perhaps they might compare their table with our Table 10 below.

Table 10 Number of Proper Factors of *N* and Product of Proper Factors of *N*					
Number of Proper Factors of *N*	1	2	3	4	5
Product of Proper Factors as a Power of *N*	N^0	$N^{1/2}$	N^1	$N^{3/2}$	N^2
Product of Proper Factors as a Power of *N* (See Note)	$N^{0/2}$	$N^{1/2}$	$N^{2/2}$	$N^{3/2}$	$N^{4/2}$

Note. In this row, the numerators of the powers of N are 0, 1, 2, 3, and 4.

In Tables 03 and 04, **no** natural number has exactly six proper factors, so we looked for them in Appendix 01 Proper Factors of Numbers 1 to 300.

- Proper factors of 64: 1, 2, 4, 8, 16, 32. Six proper factors.

We noticed that 64 is the 6th power of prime number 2 ($64 = 2 * 2 * 2 * 2 * 2 * 2 = 2^6$). "Aha!" we exclaimed hopefully. Let's try 6th powers of prime numbers 3, 5, and 7.

$3^6 = 729$ $5^6 = 15{,}625$ $7^6 = 117{,}649$

Table 11 Natural Numbers That Have Exactly six Proper Factors				
N	PFof *N*	# PF	Product of PF	N^{power}
64	1, 2, 4, 8, 16, 32	6	1 * 2 * 4 * 8 * 16 * 32 = 32,768	$64^{5/2}$
729	1, 3, 9, 27, 81, 243	6	1 * 3 * 9 * 27 * 81 * 243 = 14,348.907	$729^{5/2}$
15,625	1, 5, 25, 125, 625, 3125	6	1 * 5 * 25 * 125 * 625 * 3125 = 30,517,578,125 Oops. See note below.	$15625^{5/2}$
117,649	1, 7, 49, 343, 2401, 16,807	6	1 * 7 * 49 * 343 * 2401 * 16807 = 4,747,561,509,943. See note below.	$117649^{5/2}$

Note. Our TI-84 cannot calculate these products of exactly, so we used **Wolfram Alpha**. See the section Wolfram Alpha: a Great Teacher's Problem-Solving Tool/Toy or go directly to Wolfram Alpha http://www.wolframalpha.com/.

Observations and Conjectures:

- Observation: Every natural number *N* in Table 11 has exactly six proper factors.
- Observation: Every natural number *N* in Table 11 is a square number.

$64 = 8^2$ $729 = 81^2$ $15{,}625 = 625^2$ $117{,}649 = 343^2$

- Observation: Every natural number in Table 11 is the 6th power of a prime number.

$64 = 2^6$ $729 = 3^6$ $15{,}625 = 5^6$ $117{,}649 = 7^6$

- Conjecture: If a natural number *N* has exactly six proper factors, then the product of the proper factors of *N* is equal to the 6th power of a prime number.
- Conjecture: If a natural number *N* is the 6th power of a prime number, then *N* has exactly six proper factors.
- Conjecture: If a natural number *N* is the 6th power of a prime number, then the product of proper factors of *N* is equal to $N^{5/2}$.

Tables 03 and 04 list proper factors of numbers 2 to 50. Three numbers from 2 to 50 each have seven proper factors, shown in Table 12. Your investigators can find more by crunching numbers (preferred way) or by browsing Appendix 01 Proper Factors of Numbers 1 to 300 (lazy way).

Table 12 Natural Numbers That Have Exactly Seven Proper Factors

N	PFof *N*	# PF	Product of PF	*N* to a power
24	1, 2, 3, 4, 6, 8, 12	7	1 * 2 * 3 * 4 * 6 * 8 * 12 = 13,824	24^3
30	1, 2, 3, 5, 6, 10, 15	7	1 * 2 * 3 * 5 * 6 * 10 * 15 = 27,000	30^3
40	1, 2, 4, 5, 8, 10, 20	7	1 * 2 * 4 * 5 * 8 * 10 * 20 = 64,000	40^3

Observations and conjectures. **Beware**: conjectures can be true or false.

- Observation: Every natural number *N* in Table 12 has exactly seven proper factors.
- Observation: Every natural number *N* in Table 12 is a composite number.
- Observation: No number in Table 12 is a square number.
- Observation: For every natural number *N* in Table 12, the product of the proper factors of *N* is equal to the 3rd power of *N*. [= N^3. Also called "the cube of *N*" or "*N* cubed."]
- Conjecture: If a natural number *N* has exactly seven proper factors, then the product of the proper factors of *N* is equal to the 3rd power of *N*. Product of PF = N^3.
- Conjecture: If the product of proper factors of a natural number *N* is equal to N^3, then *N* has exactly seven proper factors.

Ahoy Teacher, We imagine you assigning investigation topics to teams of students. Some teams investigate natural numbers that have exactly seven proper factors – perhaps supplied by you or gleaned from Appendix 01 or – best! – discovered by your teams. They discover that each such number is (or is not) the cube of a natural number.

Other teams investigate natural numbers that are 3rd powers of a natural number (N^3) and discover that each such number has exactly seven proper factors. Or not.

It might happen that some of our conjectures are false. We will love it your students show that a conjecture is false, perhaps by showing a counter-example. http://en.wikipedia.org/wiki/Counterexample

If a conjecture is not true for **all** natural numbers, might it be true for a subset of natural numbers?

In Tables 03 and 04, 36 is the only natural number that has exactly eight proper factors, so we browsed Appendix 01 Proper Factors of Numbers 1 to 300, found three more numbers with exactly eight proper factors, and put all four numbers in Table 13.

Table 13 Natural Numbers That Have Exactly Eight Proper Factors

N	PFof *N*	# PF	Product of PF	N^{power}
36	1, 2, 3, 4, 6, 9, 12, 18	8	1 * 2 * 3 * 4 * 6 * 9 * 12 * 18 = 279,936	$= 36^{7/2}$
100	1, 2, 4, 5, 10, 20, 25, 50	8	1 * 2 * 4 * 5 * 10 * 20 * 25 * 50 = 10,000,000	$= 100^{7/2}$
196	1, 2, 4, 7, 14, 28, 49, 98	8	1 * 2 * 4 * 7 * 14 * 28 * 49 * 98 = 105,413,504	$= 196^{7/2}$
256	1, 2, 4, 8, 16, 32, 64, 128	8	1 * 2 * 4 * 8 * 16 * 32 * 64 * 128 = 268,435,456	$= 256^{7/2}$

Observations and conjectures:

- Observation: Every natural number in Table 13 has exactly eight proper factors.
- Observation: Every natural number in Table 13 is a composite number.
- Observation: Every natural number in Table 13 is the square of an even natural number.

  $36 = 6^2$ $100 = 10^2$ $196 = 14^2$ $256 = 16^2$

- Conjecture: If *N* has exactly eight proper factors, then *N* is a square number.
- Conjecture: If *N* is a square number, then *N* has exactly eight proper factors. Beware! Is this conjecture true? Can your investigators find a counterexample?

We searched for a pattern and noticed:

- $36 = (2 * 3)^2$, $100 = (2 * 5)^2$, $196 = (2 * 7)^2$, $256 = (2 * 8)^2$

We think we've got it! What do you think? We suggest that your investigators investigate numbers that are squares of (2 times a prime number) such as $(2 * 11)^2$, $(2 * 13)^2$, et cetera, et cetera. Note that $(2 * 8)^2$ is not the square of 2 times a prime number because 8 is not a prime number.

This investigation might lead to the following conjectures. Are they true?

- Conjecture: If a natural number *N* is equal to the square of (2 times a prime number), then *N* has exactly eight proper factors.
- Conjecture: If a natural number *N* has exactly eight proper factors, then *N* is equal to the square of (2 times a prime number).

256 = $(2 * 8)^2$ and 8 is not a prime number. Although 256 is not the square of 2 times a prime number, it does fit nicely into another pattern, shown in Table 14.

Table 14 Proper Factors and Products of Proper Factors of Numbers That Are Powers of 2

N	Proper Factors of *N*	# PF	Product of Proper Factors	N^{power}
2	1	1	1	N^{0}
4	1, 2	2	1 * 2 = 2	$N^{1/2}$
8	1, 2, 4	3	1 * 2 * 4 = 8	N^{1}
16	1, 2, 4, 8	4	1 * 2 * 4 * 8 = 64	$N^{3/2}$
32	1, 2, 4, 8, 16	5	1 * 2 * 4 * 8 * 16 = 1024	N^{2}
64	1, 2, 4, 8, 16, 32	6	1 * 2 * 4 * 8 * 16 * 32 = 32,768	$N^{5/2}$
128	1, 2, 4, 8, 16, 32, 64	7	1 * 2 * 4 * 8 * 16 * 32 * 64 = 2,097,152	N^{3}
256	1, 2, 4, 8, 16, 32, 64, 128	8	1 * 2 * 4 * 8 * 16 * 32 * 64 * 128 = 268,435,456	$N^{7/2}$

Our TI-84 is smoking after crunching all those numbers. Observations and conjectures:

- Observation: Every number in Table 14 is a power of 2.
  $2 = 2^1$, $4 = 2^2$, $8 = 2^3$, $16 = 2^4$, $32 = 2^5$, $64 = 2^6$, $128 = 2^7$, and $256 = 2^8$.
- Observation: In Table 14, odd powers of 2 have an odd number of proper factors and even powers of 2 have an even number of proper factors.
- Conjecture: If a natural number *N* is an odd power of 2, then *N* has an odd number of proper factors.
- Conjecture: If a natural number *N* is an even power of 2, then *N* has an even number of proper factors.
- Far out conjecture: If a natural number *N* is an even power of a prime number, then *N* has an even number of proper factors.
- Far out conjecture: A natural number that has an even number of proper factors is an even power of a prime number. $(\text{prime number})^2$, $(\text{prime number})^4$, $(\text{prime number})^6$, et cetera.

⚐⚐⚐	**Ahoy Teacher**, Your investigators can prove that a conjecture is false by finding one counter example: • Counter example http://en.wikipedia.org/wiki/Counterexample Proving a conjecture is true might require heavy mathemagical lifting!	⚐⚐⚐

Hmmm . . . 2 is a prime number. What about odd and even powers of other prime numbers? Powers of prime number 3 lurk in Table 15.

- Powers of 3: $3^1 = 3$ $3^2 = 9$ $3^3 = 27$ $3^4 = 81$ $3^5 = 243$ $3^6 = 729$

Table 15 Proper Factors and Products of Proper Factors of Numbers That Are Powers of 3

N	Proper Factors of *N*	# PF	Product of Proper Factors	N^{power}
3	1	1	1	N^0
9	1, 3	2	1 * 3 = 3	$N^{1/2}$
27	1, 3, 9	3	1 * 3 * 9 = 27	N^1
81	1, 3, 9, 27	4	1 * 3 * 9 * 27 = 729	$N^{3/2}$
243	1, 3, 9, 27, 81	5	1 * 3 * 9 * 27 * 81 = 59,049	N^2
729	1, 3, 9, 27, 81, 243	6	1 * 3 * 9 * 27 * 81 * 243 = 14,348,907	$N^{5/2}$

Observations and conjectures:

- Observation: Every number in Table 15 is a power of 3.
- Observation: In Table 15, odd powers of 3 (3, 27, and 243) have an odd number of proper factors and even powers of 3 (9, 81, and 729) have an even number of proper factors.
- Conjecture: If a natural number *N* is an odd power of 3, then *N* has an odd number of proper factors.
- Conjecture: If a natural number *N* is an even power of 3, then *N* has an even number of proper factors.
- Far out conjecture: If a natural number *N* is an odd power of a prime number, then *N* has an odd number of proper factors.
- Far out conjecture: If a natural number *N* is an even power of a prime number, then *N* has an even number of proper factors.
- Far out conjecture: A natural number that has an odd number of proper factors is an odd power of a prime number. $(\text{prime number})^1$, $(\text{prime number})^3$, $(\text{prime number})^5$, et cetera.
- Far out conjecture: A natural number that has an even number of proper factors is an even power of a prime number. $(\text{prime number})^2$, $(\text{prime number})^4$, $(\text{prime number})^6$, et cetera.

> **Ahoy Teacher**, Your explorers/investigators can prove that a conjecture is false by finding one counter example:
>
> - Counter example http://en.wikipedia.org/wiki/Counterexample

The End – of the Beginning – of This Exploration/Investigation | TOC

This is the end of the beginning of our exploration of products of proper factors of natural numbers. We hope that you and your teams of explorers/investigators continue the exploration.

	Wikipedia http://en.wikipedia.org/wiki/Janus : **Janus** is the god of beginnings and transitions, usually depicted as having two faces, one looking to the past and one looking to the future. January is named in his honor. We look back and think – oops, probably mistrakes, errrors, and omisions. We look forward and hope that explorers/investigators unknown to us will continue the exploration. We will too.	

We will love it if your explorers/investigators conjecture that the product of factors of a natural number N that has exactly k proper factors is equal to

- $N^{(k-1)/2}$

Amazing (if true)! The product of proper factors of a natural number is a **function** of two variables: the number itself (N) and the number of proper factors of the number (k).

Conjecture: The product of proper factors of a number N that has k proper factors is a function of N and k:

$$N^{(k-1)/2}$$

???	If the product of proper factors of a natural number $N = N^{(k-1)/2}$, what is the product of all of the factors of N (including N) equal to?	**???**

Table 16 summarizes information that your investigators probably know. They can complete Table 17 (k = 9 to 16) and then explore beyond Table 17: k = 17, 18, 19, et cetera, et cetera..

Table 16 Product of Proper Factors of *N* as a Function of *N* and *k*

Number of PF (k)	1	2	3	4	5	6	7	8
Product of PF	N^0	$N^{1/2}$	N^1	$N^{3/2}$	N^2	$N^{5/2}$	N^3	$N^{7/2}$
Product of PF	$N^{0/2}$	$N^{1/2}$	$N^{2/2}$	$N^{3/2}$	$N^{4/2}$	$N^{5/2}$	$N^{6/2}$	$N^{7/2}$

Table 17 Product of Proper Factors of *N* as a Function of *N*

Number of PF (k)	9	10	11	12	13	14	15	16
Product of PF								
Product of PF								

Wolfram Alpha: a Great Teacher's Problem-Solving Tool/Toy | TOC

As your explorers explore values of N greater than 300, they will encounter products of proper factors that are very large numbers. We began exploring numbers greater than 300 in Table 18 and soon ran into a product of proper factors that our TI-84 could not calculate exactly.

Table 18 Proper Factors and Products of Proper Factors of Numbers Greater than 300

N	Proper Factors of N	# PF	Product of Proper Factors	N^{power}
301	1, 7, 43	3	1 * 7 * 43 = 301	N^1
302	1, 2, 151	3	1 * 2 * 151 = 302	N^1
303	1, 3, 101	3	1 * 3 * 101 = 303	N^1
304	1, 2, 4, 8, 16, 19, 38, 76, 152	9	1 * 2 * 4 * 8 * 16 * 19 * 38 * 76 * 152 = 8,540,717,056	N^4
305	1, 5, 61	3	1 * 5 * 61 = 305	N^1
306	1, 2, 3, 6, 9, 17, 18, 34, 51, 102, 153	11	1 * 2 * 3 * 6 * 9 * 17 * 18 * 34 * 51 * 102 * 153 = 2.682916352E12 **Oops – see below**.	N^5 ?

Alas, our beloved TI-84 cannot calculate and display the product of proper fractions of 306 as an exact integer. Instead, it displays a 10-digit approximation in scientific notation:

- 2.682916352E12 (TI-84), also known as $2.682916352 \times 10^{12}$.

	Wolfram Alpha to the rescue! http://www.wolframalpha.com/. Wolfram Alpha can do exact arithmetic with numbers that have many digits and produce answers that have many digits!	

We copied the expression 1 * 2 * 3 * 6 * 9 * 17 * 18 * 34 * 51 * 102 * 153 to the clipboard, went to Wolfram Alpha, pasted the expression into WA's calculation box, pressed the [=] button and saw the exact 13-digit product

- 2 682 916 351 776

Wolfram Alpha uses spaces instead of commas to separate big numbers into groups of three digits from the right end of the number.

306 has 11 proper factors, so $k = 11$. Is 2 282 916 351 776 equal to $306^{(11-1)/2}$? We entered 306^((11 – 1)/2) into the WA calculation box, pressed [=] and saw:

- 2 682 916 351 776

Yup, the product of proper factors of 306 is equal to $306^{(11-1)/2} = 306^5$.

Hey! Many-digit arithmetic is just the tip of the Wolfram Alpha iceberg. You can find the proper factors of a number at Wolfram Alpha simply by asking for the proper factors of the number.

We used Wolfram Alpha to find the proper factors of numbers from 306 to 312, and then skipped down to 360 (Table 19). 360 has 23 proper factors! Column 2 of Table 19 shows what we entered into Wolfram Alpha's calculation box to get the proper factors of *N*.

Table 19 Proper Factors and Products of Proper Factors of Numbers Greater than 300

N	**Enter at Wolfram Alpha**	**Proper Factors of *N***	**# PF**
306	proper factors of 306 [=]	1, 2, 3, 6, 9, 17, 18, 34, 51, 102, 153	11
307	proper factors of 307 [=]	1	1
308	proper factors of 308 [=]	1, 2, 4, 7, 11, 14, 22, 28, 44, 77, 154	11
309	proper factors of 309 [=]	1, 3, 103	3
310	proper factors of 310 [=]	1, 2, 5, 10, 31, 62, 155	7
311	proper factors of 311 [=]	1	1
312	proper factors of 312 [=]	1, 2, 3, 4, 6, 8, 12, 13, 24, 26, 39, 52, 78, 104, 156	15
. . .	skip to 360	, , ,	
360	proper factors of 360 [=]	1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 15, 18, 20, 24, 30, 36, 40, 45, 60, 72, 90, 120, 180	23

Thanks to Wolfram Alpha, we and you and your students can quickly and easily find the proper factors of a number – there is probably a limit, but we do not know what it is.

As of 2014-09-01, at Wolfram Alpha you can:

- Enter **proper factors of *N***, press **[=]**, and get a list of the proper factors of *N*. You supply the value of *N*.
- Enter **sum of proper factors of *N***, press **[=]**, and get the sum of the proper factors of *N*. You supply the value of *N*. You will also see a list of the proper factors of *N*.

⚐⚐⚐	Today (2014-09-01) at Wolfram Alpha, we were unable to get the product of proper factors of 360 by entering **product of proper factors of 360** and pressing **[=]**. Wolfram Alpha is continually adding more alakazams, so by the time you read this, perhaps WA can calculate the product of proper factors of a number.	⚐⚐⚐

Questions That Occurred to Us Along the Trail | TOC

While we were on the trail – exploring products of proper factors – questions about number of proper factors and sums of proper factors occurred to us. In order to stay on task (products of proper factors), we recorded our questions in this section in order to remember them and to share them with you. We hope that some of these questions might be interesting topics for your explorers/investigators to explore/investigate.

Question 01. The sum of proper factors of 6 is equal to the product of proper factors of 6.

- 1 + 2 + 3 = 6 and 1 * 2 * 3 = 6

Is there another natural number for which the sum of its proper factors is equal to the product of its proper factors? [We don't know the answer.]

Question 02. The first (least) natural number that has exactly one proper factor is 2. The one-and-only proper factor of 2 is 1.

- What is the first (least) natural number that has exactly two proper factors?
- What is the first (least) natural number that has exactly three proper factors?
- What is the first (least) natural number that has exactly four proper factors?
- What is the first (least) natural number that has exactly five proper factors?
- What is the first (least) natural number that has exactly six proper factors?
- What is the first (least) natural number that has exactly seven proper factors?
- What is the first (least) natural number that has exactly eight proper factors?
- What is the first (least) natural number that has exactly nine proper factors?
- What is the first (least) natural number that has exactly 10 proper factors?

Et cetera, et cetera. This leads up to the questions:

- What is the first (least) natural number that has exactly *k* proper factors, where you and your explorers/investigators choose the value of *k*?
- For any number *k* of proper factors that you or your explorers/investigators choose, does there exist a natural number *N* that has *k* proper factors?
- For any number *k* that you or your explorers/investigators choose, can you or they **construct** a natural number *N* that has exactly *k* proper factors? Hey! We like this one.

⚐⚐⚐	Peruse Appendix 01 Proper Factors of Numbers 1 to 300 to try to find answers. If you can't find answers in Appendix 01, **go beyond**! Investigate numbers greater than 300. Go where we have not explored. Engage!	⚐⚐⚐

Question 03. The first (least) natural number *N* for which the sum of proper factors of *N* is less than *N* is 2.

- Proper factor of 2: 1
- Sum of proper factors of 2 = 1. 1 < 2. 2 is a deficient number.
- What is the first natural number *N* **that is not a prime number** for which the sum of proper factors of *N* is less than *N*?
- Et cetera, et cetera. Does this go on forever? Is there an ‘infinite’ number of natural numbers *N* that are not prime numbers and have a sum of proper factors less than *N*? Or only a finite number? If only a finite number, then what is the greatest number *N* that is not a prime number and has a sum of proper factors less than *N*?

Question 04. The first (least) natural number *N* for which the sum of proper factors of *N* is greater than *N* is 12.

- Proper factors of 12: 1, 2, 3, 4, and 6.
- Sum of proper factors of 12 = 16. 16 > 12. 12 is an abundant number.
- What is the second natural number *N* for which the sum of proper factors of *N* is greater than *N*?
- Et cetera, et cetera. Does this go on forever? Is there an ‘infinite’ number of natural numbers *N* that have a sum of proper factors greater than *N*? Or only a finite number? If there are only a finite number, then what is the greatest number *N* that has a sum of proper factors greater than *N*?

Question 05. The first (least) natural number *N* for which the sum of proper factors of *N* is equal to *N* is 6.

- Proper factors of 6: 1, 2, and 3.
- Sum of proper factors of 6 = 1 + 2 + 3 = 6. 6 is the first perfect number.
- What is the second natural number *N* for which the sum of proper factors of *N* is equal to *N*?
- Et cetera, et cetera. Does this go on forever? Is there an ‘infinite’ number of natural numbers *N* that have a sum of proper factors equal to *N*? Or only a finite number? If there are only a finite number, then what is the greatest number *N* that has a sum of proper factors equal to *N*?

Question 06. In the set of natural numbers 1 to 100, how many numbers have 1 proper factor? How many numbers have 2 proper factors? How many numbers have 3 proper factors? Et cetera, et cetera. Make a **frequency table** showing the number of proper factors (# PF) and the frequency of occurrence of that number of proper factors in the set of natural numbers 1 to 100. See http://en.wikipedia.org/wiki/Frequency_distribution

# PF	0	1	2	3	4	5	6	7	8	10
Frequency										

# PF	10	11	12	13	14	15	16	17	18	20
Frequency										

Enough. It is time to proofread this eBook (groan – lotsa work!) and post it in our free download page at IAE Pedia: http://i-a-e.org/downloads/cat_view/86-free-ebooks-by-bob-albrecht.html

Farewell. May dragons of good fortune be with you and your explorers/investigators as you explore, investigate, and go adventuring in proper factor land – or elsewhere.

Play Together, Learn Together

DragonFun image by Marcie Hawthorne http://marciehawthorne.com/

Appendix 01 Proper Factors of Numbers 1 to 300 | TOC

Ahoy Teacher, This appendix is for **You**. We hope that you show only a wee bit to your students AFTER they have found the number of proper factors of a bunch of numbers. We will love it if they find errrors, mistrakes, and oopses in our tables and tell us about it by sending email to alakazam@ssgaia.com.

Table App01-01 Proper Factors of Numbers 1 to 50				
N	**Proper Factors of *N***		***N***	**Proper Factors of *N***
1	none		26	1, 2, 13
2	1		27	1, 3, 9
3	1		28	1, 2, 4, 7, 14
4	1, 2		29	1
5	1		30	1, 2, 3, 5, 6, 10, 15
6	1, 2, 3		31	1
7	7		32	1, 2, 4, 8, 16
8	1, 2, 4		33	1, 3, 11
9	1, 3		34	1, 2, 17
10	1, 2, 5		35	1, 5, 7
11	1		36	1, 2, 3, 4, 6, 9, 12, 18
12	1, 2, 3, 4, 6		37	1
13	1		38	1, 2, 19
14	1, 2, 7		39	1, 3, 13
15	1, 3, 5		40	1, 2, 4, 5, 8, 10, 20
16	1, 2, 4, 8		41	1
17	1		42	1, 2, 3, 6, 7, 14, 21
18	1, 2, 3, 6, 9		43	1
19	1		44	1, 2, 4, 11, 22
20	1, 2, 4, 5, 10		45	1, 3, 5, 9, 15
21	1, 3, 7		46	1, 2, 23
22	1, 2, 11		47	1
23	1		48	1, 2, 3, 4, 6, 8, 12, 16, 24
24	1, 2, 3, 4, 6, 8, 12		49	1, 7
25	1, 5		50	1, 2, 5, 10, 25

Table App01-02 Proper Factors of Numbers 51 to 100				
N	**Proper Factors of *N***		***N***	**Proper Factors of *N***
51	1, 3, 17		76	1, 2, 4, 19, 38
52	1, 2, 4, 13, 26		77	1, 7, 11
53	1		78	1, 2, 3, 6, 13, 26, 39
54	1, 2, 3, 6, 9, 18, 27		79	1
55	1, 5, 11		80	1, 2, 4, 5, 8, 10, 16, 20, 40
56	1, 2, 4, 7, 8, 14, 28		81	1, 3, 9, 27
57	1, 3, 19		82	1, 2, 41
58	1, 2, 29		83	1
59	1		84	1, 2, 3, 4, 6, 7, 12, 14, 21, 28, 42
60	1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30		85	1, 5, 17
61	1		86	1, 2, 43
62	1, 2, 31		87	1, 3, 29
63	1, 3, 7, 9, 21		88	1, 2, 4, 8, 11, 22, 44
64	1, 2, 4, 8, 16, 32		89	1
65	1, 5, 13		90	1, 2, 3, 5, 6, 9, 10, 15, 18, 30, 45
66	1, 2, 3, 6, 11, 22, 33		91	1, 7, 13
67	1		92	1, 2, 4, 23, 46
68	1, 2, 4, 17, 34		93	1, 3, 31
69	1, 3, 23		94	1, 2, 47
70	1, 2, 5, 7, 10, 14, 35		95	1, 5, 19
71	1		96	1, 2, 3, 4, 6, 8, 12, 16, 24, 32, 48
72	1, 2, 3, 4, 6, 8, 9, 12, 18, 24, 36		97	1
73	1		98	1, 2, 7, 14, 49
74	1, 2, 37		99	1, 3, 9, 11, 33
75	1, 3, 5, 15, 25		100	1, 2, 4, 5, 10, 20, 25, 50

Table App01-03 Proper Factors of Numbers 101 to 150				
N	**Proper Factors of *N***		***N***	**Proper Factors of *N***
101	1		126	1, 2, 3, 6, 7, 9, 14, 18, 21, 42, 63
102	1, 2, 3, 6, 17, 34, 51		127	1
103	1		128	1, 2, 4, 8, 16, 32, 64
104	1, 2, 4, 8, 13, 26, 52		129	1, 3, 43
105	1, 3, 5, 7, 15, 21, 35		130	1, 2, 5, 10, 13, 26, 65
106	1, 2, 53		131	1
107	1		132	1, 2, 3, 4, 6, 11, 12, 22, 33, 44, 66
108	1, 2, 3, 4, 6, 9, 12, 18, 27, 36, 54		133	1, 7, 19
109	1		134	1, 2, 67
110	1, 2, 5 10, 11, 22, 55		135	1, 3, 5, 9, 15, 27, 45
111	1, 3, 37		136	1, 2, 4, 8, 17, 34, 68
112	1, 2, 4, 7, 8, 14, 16, 28, 56		137	1
113	1		138	1, 2, 3, 6, 23, 46, 69
114	1, 2, 3, 6, 19, 38, 57		139	1
115	1, 5, 23		140	1, 2, 4, 5, 7, 10, 14, 20, 28, 35, 70
116	1, 2, 4, 29, 58		141	1, 3, 47
117	1, 3, 9, 13, 39		142	1, 2, 71
118	1, 2, 59		143	1, 11, 13
119	1, 7, 17		144	1, 2, 3, 4, 6, 8, 9, 12, 16, 18, 24, 36, 48, 72
120	1, 2, 3, 4, 5, 6, 8, 10, 12, 15, 20, 24, 30, 40, 60		145	1, 5, 29
121	1, 11		146	1, 2, 73
122	1, 2, 61		147	1, 3, 7, 21, 49
123	1, 3, 41		148	1, 2, 4, 37, 74
124	1, 2, 4, 31, 62		149	1
125	1, 5, 25		150	1, 2, 3, 5, 6, 10, 15, 25, 30, 50, 75

Table App01-04 Proper Factors of Numbers 151 to 200				
N	**Proper Factors of *N***		***N***	**Proper Factors of *N***
151	1		176	1, 2, 4, 8, 11, 16, 22, 44, 88
152	1, 2, 4, 8, 19, 38, 76		177	1, 3, 59
153	1, 3, 9, 17, 51		178	1, 2, 89
154	1, 2, 7, 11, 14, 22, 77		179	1
155	1, 5, 31		180	1, 2, 3, 4, 5, 6, 9, 10, 12, 15, 18, 20, 30, 36, 45, 60, 90
156	1, 2, 3, 4, 6, 12, 13, 26, 39, 52, 78		181	1
157	1		182	1, 2, 7, 13, 14, 26, 91
158	1, 2, 79		183	1, 3, 61
159	1, 3, 53		184	1, 2, 4, 8, 23, 46, 92
160	1, 2, 4, 5, 8, 10, 16, 20, 32, 40, 80		185	1, 5, 37
161	1, 7, 23		186	1, 2, 3, 6, 31, 62, 93
162	1, 2, 3, 6, 9, 18, 27, 54, 81		187	1, 11, 17
163	1		188	1, 2, 4, 47, 94
164	1, 2, 4, 41, 82		189	1, 3, 7, 9, 21, 27, 63
165	1, 3, 5, 11, 15, 33, 55		190	1, 2, 5, 10, 19, 38, 95
166	1, 2, 83		191	1
167	1		192	1, 2, 3, 4, 6, 8, 12, 16, 24, 32, 48, 64, 96
168	1, 2, 3, 4, 6, 7, 8, 12, 14, 21, 24, 28, 42, 56, 84		193	1
169	1, 13		194	1, 2, 97
170	1, 2, 5, 10, 17, 34, 85		195	1, 3, 5, 13, 15, 39, 65
171	1, 3, 9, 19, 57		196	1, 2, 4, 7, 14, 28, 49, 98
172	1, 2, 4, 43, 86		197	1
173	1		198	1, 2, 3, 6, 9, 11, 18, 22, 33, 66, 99
174	1, 2, 3, 6, 29, 58, 87		199	1
175	1, 5, 7, 25, 35		200	1, 2, 4, 5, 8, 10, 20, 25, 40, 50, 100

Table App01-05 Proper Factors of Numbers 201 to 250				
N	**Proper Factors of *N***		***N***	**Proper Factors of *N***
201	1, 3, 67		226	1, 2, 113
202	1, 2, 101		227	1
203	1, 7, 29		228	1, 2, 3, 4, 6, 12, 19, 38, 57, 76, 114
204	1, 2, 3, 4, 6, 12, 17, 34, 51, 68, 102		229	1
205	1, 5, 41		230	1, 2, 5, 10, 23, 46, 115
206	1, 2, 103		231	1, 3, 7, 11, 21, 33, 77
207	1, 3, 9, 23, 69		232	1, 2, 4, 8, 29, 58, 116
208	1, 2, 4, 8, 13, 16, 26, 52, 104		233	1
209	1, 11, 19		234	1, 2, 3, 6, 9, 13, 18, 26, 39, 78, 117
210	1, 2, 3, 5, 6, 7, 10, 14, 15, 21, 30, 35, 42, 70, 105		235	1, 5, 47
211	1		236	1, 2, 4, 59, 118
212	1, 2, 4, 53, 106		237	1, 3, 79
213	1, 3, 71		238	1, 2, 7, 14, 17, 34, 119
214	1, 2, 107		239	1
215	1, 5, 43		240	1, 2, 3, 4, 5, 6, 8, 10, 12, 15, 16, 20, 24, 30, 40, 48, 60, 80, 120
216	1, 2, 3, 4, 6, 8, 9, 12, 18, 24, 27, 36, 54, 72, 108		241	1
217	1, 7, 31		242	1, 2, 11, 22, 121
218	1, 2, 109		243	1, 3, 9, 27, 81
219	1, 3, 73		244	1, 2, 4, 61, 122
220	1, 2, 4, 5, 10, 11, 20, 22, 44, 55, 110		245	1, 5, 7, 35, 49
221	1, 13, 17		246	1, 2, 3, 6, 41, 82, 123
222	1, 2, 3, 6, 37, 74, 111		247	1, 13, 19
223	1		248	1, 2, 4, 8, 31, 62, 124
224	1, 2, 4, 7, 8, 14, 16, 28, 32, 56, 112		249	1, 3, 83
225	1, 3, 5, 9, 15, 25, 45, 75		250	1, 2, 5, 10, 25, 50, 125

Table App01-06 Proper Factors of Numbers 251 to 300				
N	**Proper Factors of *N***		***N***	**Proper Factors of *N***
251	1		276	1, 2, 3, 4, 6, 12, 23, 46, 69, 92, 138
252	1, 2, 3, 4, 6, 7, 9, 12, 14, 18, 21, 28, 36, 42, 63, 84, 126		277	1
253	1, 11, 23		278	1, 2, 139
254	1, 2, 127		279	1, 3, 9, 31, 93
255	1, 3, 5, 15, 17, 51, 85		280	1, 2, 4, 5, 7, 8, 10, 14, 20, 28, 35, 40, 56, 70, 140
256	1, 2, 4, 8, 16, 32, 64, 128		281	1
257	1		282	1, 2, 3, 6, 47, 94, 141
258	1, 2, 3, 6, 43, 86, 129		283	1
259	1, 7, 37		284	1, 2, 4, 71, 142
260	1, 2, 4, 5, 10, 13, 20, 26, 52, 65, 130		285	1, 3, 5, 15, 19, 57, 95
261	1, 3, 9, 29, 87		286	1, 2, 11, 13, 22, 26, 143
262	1, 2, 131		287	1, 7, 41
263	1		288	1, 2, 3, 4, 6, 8, 9, 12, 16, 18, 24, 32, 36, 48, 72, 96, 144
264	1, 2, 3, 4, 6, 8, 11, 12, 22, 24, 33, 44, 66, 88, 132		289	1, 17
265	1, 5, 53		290	1, 2, 5, 10, 29, 58, 145
266	1, 2, 7, 14, 19, 38, 133		291	1, 3, 97
267	1, 3, 89		292	1, 2, 4, 73, 146
268	1, 2, 4, 67, 134		293	1
269	1		294	1, 2, 3, 6, 7, 14, 21, 42, 49, 98, 147
270	1, 2, 3, 5, 6, 9, 10, 15, 18, 27, 30, 45, 54, 90, 135		295	1, 5, 59
271	1		296	1, 2, 4, 8, 37, 74, 148
272	1, 2, 4, 8, 16, 17, 34, 68, 136		297	1, 3, 9, 11, 27, 33, 99
273	1, 3, 7, 13, 21, 39, 91		298	1, 2, 149
274	1, 2, 137		299	1, 13, 23
275	1, 5, 11, 25, 55		300	1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 25, 30, 50, 60, 75, 100,150

Appendix 02 Number & Sum of Proper Factors of Numbers 1 to 300 | TOC

Ahoy Teacher, This appendix is for YOU. We hope that you show only a wee bit to your students **AFTER** they have found the number and sum of proper factors of a bunch of numbers. We will love it if they find errrors, mistrakes, and oopses in these tables our tables and tell us about it by sending email to alakazam@ssgaia.com.

Table App02-01 Number and Sum of Proper Factors of Numbers 1 to 50						
N	**Number of PF**	**Sum of PF**		***N***	**Number of PF**	**Sum of PF**
1	none	none		26	3	16
2	1	1		27	3	13
3	1	1		28	5	28
4	2	3		29	1	1
5	1	1		30	7	42
6	3	6		31	1	1
7	1	1		32	5	31
8	3	7		33	3	15
9	2	4		34	3	20
10	3	8		35	3	13
11	1	1		36	8	55
12	5	16		37	1	1
13	1	1		38	3	22
14	3	10		39	3	17
15	3	9		40	7	50
16	4	15		41	1	1
17	1	1		42	7	54
18	5	21		43	1	1
19	1	1		44	5	40
20	5	22		45	5	33
21	3	11		46	3	26
22	3	14		47	1	1
23	1	1		48	9	76
24	7	36		49	2	8
25	2	6		50	5	43

Table App02-02 Number and Sum of Proper Factors of Numbers 51 to 100						
N	**Number of PF**	**Sum of PF**		***N***	**Number of PF**	**Sum of PF**
51	3	21		76	5	64
52	5	46		77	3	19
53	1	1		78	7	90
54	7	66		79	1	1
55	3	17		80	9	106
56	7	64		81	4	40
57	3	23		82	3	44
58	3	32		83	1	1
59	1	1		84	11	140
60	11	108		85	3	23
61	1	1		86	3	46
62	3	34		87	3	33
63	5	41		88	7	92
64	6	63		89	1	1
65	3	19		90	11	144
66	7	78		91	3	21
67	1	1		92	5	76
68	5	58		93	3	35
69	3	27		94	3	50
70	7	74		95	3	25
71	1	1		96	11	156
72	11	123		97	1	1
73	1	1		98	5	73
74	3	40		99	5	57
75	5	49		100	8	117

Table App02-03 Number and Sum of Proper Factors of Numbers 101 to 150							
N	**Number of PF**	**Sum of PF**		***N***	**Number of PF**	**Sum of PF**
101	1	1		126	11	186
102	7	114		127	1	1
103	1	1		128	7	127
104	7	106		129	3	47
105	7	87		130	7	122
106	3	56		131	1	1
107	1	1		132	11	204
108	11	172		133	3	27
109	1	1		134	3	70
110	7	106		135	7	105
111	3	41		136	7	134
112	9	136		137	1	1
113	1	1		138	7	150
114	7	126		139	1	1
115	3	29		140	11	196
116	5	94		141	3	51
117	5	65		142	3	74
118	3	62		143	3	25
119	3	25		144	14	259
120	15	240		145	3	35
121	2	12		146	3	76
122	3	64		147	5	81
123	3	45		148	5	118
124	5	100		149	1	1
125	3	31		150	11	222

Table App02-04 Number and Sum of Proper Factors of Numbers 151 to 200						
N	**Number of PF**	**Sum of PF**		***N***	**Number of PF**	**Sum of PF**
151	1	1		176	9	196
152	7	148		177	3	63
153	5	81		178	3	92
154	7	134		179	1	1
155	3	37		180	17	366
156	11	236		181	1	1
157	1	1		182	7	154
158	3	82		183	3	65
159	3	57		184	7	176
160	11	218		185	3	43
161	3	31		186	7	198
162	9	201		187	3	29
163	1	1		188	5	148
164	5	130		189	7	131
165	7	123		190	7	170
166	3	86		191	1	1
167	1	1		192	13	316
168	15	312		193	1	1
169	2	14		194	3	100
170	7	154		195	7	141
171	5	89		196	8	203
172	5	136		197	1	1
173	1	1		198	11	270
174	7	186		199	1	1
175	5	73		200	11	265

Table App02-05 Number and Sum of Proper Factors of Numbers 201 to 250						
N	**Number of PF**	**Sum of PF**		***N***	**Number of PF**	**Sum of PF**
201	3	71		226	3	116
202	3	104		227	1	1
203	3	37		228	11	332
204	11	300		229	1	1
205	3	47		230	7	202
206	3	106		231	7	153
207	5	105		232	7	218
208	9	226		233	1	1
209	3	31		234	11	312
210	15	366		235	3	53
211	1	1		236	5	184
212	5	166		237	3	83
213	3	75		238	7	194
214	3	110		239	1	1
215	3	49		240	19	504
216	15	384		241	1	1
217	3	39		242	5	157
218	3	112		243	5	121
219	3	77		244	5	190
220	11	284		245	5	97
221	3	31		246	7	258
222	7	234		247	3	33
223	1	1		248	7	232
224	11	280		249	3	87
225	8	178		250	7	218

Table App02-06 Number and Sum of Proper Factors of Numbers 201 to 250						
N	**Number of PF**	**Sum of PF**		***N***	**Number of PF**	**Sum of PF**
251	1	1		276	11	396
252	17	476		277	1	1
253	3	35		278	3	142
254	3	130		279	5	137
255	7	177		280	15	440
256	8	255		281	1	1
257	1	1		282	7	294
258	7	270		283	1	1
259	3	45		284	5	220
260	11	328		285	7	195
261	5	129		286	7	218
262	3	134		287	3	49
263	1	1		288	17	531
264	15	456		289	2	18
265	3	59		290	7	250
266	7	214		291	3	101
267	3	93		292	5	226
268	5	208		293	1	1
269	1	1		294	11	390
270	15	450		295	3	65
271	1	1		296	7	274
272	9	286		297	7	183
273	7	175		298	3	152
274	3	140		299	3	37
275	5	97		300	17	568

Appendix 03 Products of Proper Factors of Numbers 1 to 100 | TOC

We used our TI-84 calculator to crunch the numbers, so we use the TI-84's multiplication symbol (*) in this table.

Table App03-01 Products of Proper Factors of Natural Numbers 1 to 50				
N	**Product of Proper Factors of *N***		***N***	**Product of Proper Factors of *N***
1	no proper factors		26	1 * 2 * 13 = 26
2	1		27	1 * 3 * 9 = 27
3	1		28	1 * 2 * 4 * 7 * 14 = 784
4	1 * 2 = 2		29	1
5	1		30	1 * 2 * 3 * 5 * 6 * 10 * 15 = 27,000
6	1 *2 * 3 = 6		31	1
7	1		32	1 * 2 * 4 * 8 * 16 = 31
8	1 * 2 * 4 = 8		33	1 * 3 * 11 = 33
9	1 * 3 = 3		34	1 * 2 * 17 = 34
10	1 * 2 * 5 =10		35	1 * 5 * 7 = 35
11	1		36	1 * 2 * 3 * 4 * 6 * 9 * 12 * 18 = 279,936
12	1 * 2 * 3 * 4 * 6 = 144		37	1
13	1		38	1 * 2 * 19 = 38
14	1 * 2 * 7 = 14		39	1 * 3 * 13 = 39
15	1 * 3 * 5 = 15		40	1 * 2 * 4 * 5 * 8 * 10 * 20 = 64,000
16	1 * 2 * 4 * 8 = 15		41	1
17	1		42	1 * 2 * 3 * 6 * 7 * 14 * 21 = 74,088
18	1 * 2 * 3 * 6 * 9 = 324		43	1
19	1		44	1 * 2 * 4 * 11 * 22 = 1936
20	1 * 2 * 4 * 5 * 10 = 400		45	1 * 3 * 5 * 9 * 15 = 2025
21	1 * 3 * 7 = 21		46	1 * 2 * 23 = 46
22	1 * 2 * 11 = 22		47	1
23	1		48	1*2*3*4*6*8*12*16*24 = 5,308,416
24	1 * 2 * 3 * 4 * 6 * 8 * 12 = 13,824		49	1 * 7 = 7
25	1 * 5 = 5		50	1 * 2 * 5 * 10 * 25 = 2500

We use our TI-84 calculator to crunch the numbers, so we use the TI-84's multiplication symbol (*) in this table.

Table App03-02 Products of Proper Factors of Natural Numbers 51 to 100				
N	**Product of Proper Factors of *N***		***N***	**Product of Proper Factors of *N***
51	1 * 3 * 17 = 51		76	1 * 2 * 4 * 19 * 38 = 5776
52	1 * 2 * 4 * 13 * 26 = 2704		77	1 * 7 * 11 = 77
53	1		78	1 * 2 * 3 * 6 * 13 * 26 * 39 = 474,552
54	1 * 2 * 3 * 6 * 9 * 18 * 27 = 157,464		79	1
55	1 * 5 * 11 = 55		80	1*2*4*5*8*10*16*20*40 = 40,960,000
56	1 * 2 * 4 * 7 * 8 * 14 * 28 = 175,616		81	1 * 3 * 9 * 27 = 729
57	1 * 3 * 19 = 57		82	1 * 2 * 41 = 82
58	1 * 2 * 29 = 58		83	1
59	1		84	1*2*3*4*6*7*12*14*21*28*42 = 4,182,119,424
60	1*2*3*4*5*6*10*12*15*20*30 = 777,600,000		85	1 * 5 * 17 = 85
61	1		86	1 * 2 * 43 = 86
62	1 * 2 * 31 = 62		87	1 * 3 * 29 = 87
63	1 * 3 * 7 * 9 * 21 = 3969		88	1 * 2 * 4 * 8 * 11 * 22 * 44 = 681,472
64	1 * 2 * 4 * 8 * 16 * 32 = 32,768		89	1
65	1 * 5 * 13 = 65		90	1*2*3*5*6*9*10*15*18*30*45 = 5,904,900,000
66	1 * 2 * 3 * 6 * 11 * 22 * 33 = 287,496		91	1 * 7 * 13 = 91
67	1		92	1 * 2 * 4 * 23 * 46 = 8464
68	1 * 2 * 4 * 17 * 34 = 4624		93	1 * 3 * 31 = 93
69	1 * 3 * 23 = 69		94	1 * 2 * 47 = 94
70	1 * 2 * 5 * 7 * 10 * 14 * 35 = 343,000		95	1 * 5 * 19 = 95
71	1		96	1*2*3*4*6*8*12*16*24*32*48 = 8,153,726,976
72	1*2*3*4*6*8*9*12*18*24*36 = 1,934,917,632		97	1
73	1		98	1 * 2 * 7 * 14 * 49 = 9604
74	1 * 2 * 37 = 74		99	1 * 3 * 9 * 11 * 33 = 9801
75	1 * 3 * 5 * 15 * 25 = 5625		100	1*2*4*5*10*20*25*50 = 10,000,000

Appendix 04 Conjectures in This eBook – the Whole Shebang | TOC

In this eBook, we have posed many **observations** and **conjectures**. The conjectures might be true not true, for all natural number. We collected the conjectures into this appendix.

- Observation http://en.wikipedia.org/wiki/Observation
- Conjecture http://en.wikipedia.org/wiki/Conjecture

A conjecture can be **true** or **false**. A conjecture based on the data in the tables in this eBook might be true for sets of natural numbers in those tables, but false for a different set, such as natural numbers 1 to 100 or 100 to 200, or 1000 to 1999, or *m* to *n*, where you and your explorers/investigators choose the values of *m* and *n*.

⚐⚐⚐	Your investigators can prove that a conjecture is false by finding one counter example: • Counter example http://en.wikipedia.org/wiki/Counterexample	⚐⚐⚐

Here they are, the whole shebang, all of the conjectures that we posed in this eBook. We wonder what conjectures you and your explorers/investigators will make – **conjectures that did not occur to us**. May dragons of exploration, investigation, and discovery be your companions.

Conjecture: If N is a prime number, then N has 1 proper factor.

Conjecture: If N is a prime number, then the product of proper factors of $N = 1$.

Conjecture: If N is a prime number, then the product of proper factors of $N = N^0$.

Conjecture: The square of a prime number has exactly two proper factors.

Conjecture: If a natural number N has exactly two proper factors, then the product of the proper factors of N is equal to the square root of N.

Conjecture: If a natural number N is the square of a prime number, then the product of the proper factors of N is equal to $N^{1/2}$.

Conjecture: If N is a square number, then N has an even number of proper factors.

Conjecture: If a natural number N has exactly three proper factors, then the product of the proper factors of N is equal to $N = N^1$.

Conjecture: If the product of the proper factors of a natural number N is equal to N, Then N has exactly three proper factors.

Ahoy Teacher, Your investigators can prove that a conjecture is false by finding one counter example:

- Counter example http://en.wikipedia.org/wiki/Counterexample

Conjecture: If a natural number N has exactly three proper factors, then two of the proper factors are prime numbers.

Conjecture: If a natural number N has exactly four proper factors, then N is equal to the 4th power of a prime number. N = (prime number)^4.

Conjecture: If a natural number N is equal to the 4th power of a prime number, then N has exactly four proper factors.

Conjecture: If a natural number N has exactly four proper factors, then the product of the proper factors is equal to $N^{3/2}$. [Product of proper factors of $N = N^{3/2}$]

Conjecture: If a natural number N has exactly five proper factors, then the product of proper factors of N is equal to the square of N. [Product of proper factors of $N = N^2$].

Conjecture: If the product of the proper factors of a natural number N is equal to N^2, then N has exactly three proper factors.

Conjecture: If a natural number N is a composite number and is not a square number, then N has an odd number of proper factors.

Conjecture: If a natural number N has exactly six proper factors, then the product of the proper factors of N is equal to the 6th power of a prime number.

Conjecture: If a natural number N is the 6th power of a prime number, then N has exactly six proper factors.

Conjecture: If a natural number N is the 6th power of a prime number, then the product of proper factors of N is equal to $N^{5/2}$.

Conjecture: If a natural number N has exactly seven proper factors, then the product of the proper factors of N is equal to the 3rd power of N. Product of PF = N^3.

Conjecture: If the product of proper factors of a natural number N is equal to N^3, then N has exactly seven proper factors.

Conjecture: If N has exactly eight proper factors, then N is a square number.

Conjecture: If N is a square number, then N has exactly eight proper factors. Beware! Is this conjecture true? Can your investigators find a counterexample?

Conjecture: If a natural number *N* is equal to the square of (2 times a prime number), then N has exactly eight proper factors.

Conjecture: If a natural number *N* has exactly eight proper factors, then *N* is equal to the square of (2 times a prime number).

Conjecture: If a natural number *N* is an odd power of 2, then *N* has an odd number of proper factors.

Conjecture: If a natural number *N* is an even power of 2, then *N* has an even number of proper factors.

Far out conjecture: A natural number that has an even number of proper factors is an even power of a prime number. $(\text{prime number})^2$, $(\text{prime number})^4$, $(\text{prime number})^6$, et cetera.

Far out conjecture: If a natural number *N* is an even power of a prime number, then *N* has an even number of proper factors.

Conjecture: If a natural number *N* is an odd power of 3, then *N* has an odd number of proper factors.

Conjecture: If a natural number *N* is an even power of 3, then *N* has an even number of proper factors.

Conjecture: If a natural number *N* is an odd power of a prime number, then *N* has an odd number of proper factors.

Conjecture: If a natural number *N* is an even power of a prime number, then *N* has an even number of proper factors.

Conjecture: A natural number that has an odd number of proper factors is an odd power of a prime number. $(\text{prime number})^1$, $(\text{prime number})^3$, $(\text{prime number})^5$, et cetera.

Conjecture: A natural number that has an even number of proper factors is an even power of a prime number. $(\text{prime number})^2$, $(\text{prime number})^4$, $(\text{prime number})^6$, et cetera.

Conjecture: The product of proper factors of a number *N* that has *k* proper factors is a function of *N* and *k*: Product of proper factors = $N^{(k-1)/2}$

Appendix 05 Excerpts from Mathemagical Numbers 1 to 99 | TOC

If you browse our eBook *Mathemagical Numbers 1 to 99*, you will find odd numbers, even numbers, prime numbers, emirps, palprimes, composite numbers and their prime factorizations, factors, proper factors, sums of factors, sums of proper factors, deficient numbers, perfect numbers, abundant numbers, square numbers, cubic numbers, triangular numbers, factorial numbers, Fibonacci numbers, palindromic numbers, and the number of protons in atoms with atomic numbers 1 to 99. Download *Mathemagical Numbers 1 to 99* as a PDF file or Word file at

- http://i-a-e.org/downloads/cat_view/86-free-ebooks-by-bob-albrecht.html

And now (please imagine fanfare), here are excerpts from *Mathemagical Numbers 1 to 99*.

1 (one)

1 is a natural number.
1 is the predecessor of 2.
1 does not have a predecessor.
1 is the least natural number.
1 is an odd number.
1 is the least odd number.
1 is not a prime number.
1 is not a composite number.
1 is the only natural number that has exactly 1 factor.
One and only factor of 1: 1
1 has no proper factor.
1 is a factor of every natural number.
1 is the multiplicative identity. 1 x $n = n$ and n x 1 = n.
1 is a square number. 1 = 1 x 1 = 1^2
1 is a cubic number. 1 = 1 x 1 x 1 = 1^3
1 is a power of 2. 1 = 2^0
1 is a triangular number.
1 is a factorial number. 1! = 1
1 is a Fibonacci number.
Geometry: a ray has 1 endpoint.
A hydrogen (H) atom has 1 proton.

Here are some "onederful" words from an old folk song.

I'll sing you one-o
Green grow the rushes-o
What is your one-o?
One is one and all alone
And evermore shall be so.

2 (two)

2 is a natural number.
2 is the successor of 1.
2 is the predecessor of 3.
2 is an even number.
2 is the least even number.
2 is a prime number.
2 is the least prime number.
2 is the only even prime number.
Factors of 2: 1, 2
2 is the least number that has exactly 2 different factors.
Proper factor of 2: 1
2 is the least number that has exactly 1 proper factor.
Sum of factors of 2 = 3
Sum of proper factors of 2 = 1
2 is a deficient number.
2 is a power of 2. $2 = 2^1$
2 is a factorial number. 2! = 1 x 2 = 2
2 is a Fibonacci number.
Geometry: a line segment has 2 endpoints.
A helium (He) atom has 2 protons.

3 (three)

3 is a natural number.
3 is the successor of 2.
3 is the predecessor of 4.
3 is an odd number.
3 is a prime number.
3 is the least odd prime number.
Factors of 3: 1, 3
Proper factor of 3: 1
Sum of factors of 3 = 4
Sum of proper factors of 3 = 1
3 is a deficient number.
3 is a triangular number. 3 = 1 + 2
3 is a Fibonacci number.
Geometry: a triangle has 3 sides and 3 vertices.
A lithium (Li) atom has 3 protons.

4 (four)

4 is a natural number.
4 is the successor of 3.
4 is the predecessor of 5.
4 is an even number.
4 is a composite number.

Prime factorization: 4 = 2 x 2
4 is the least composite number.
Factors of 4: 1, 2, 4
4 is the least number that has exactly 3 different factors.
Proper factors of 4: 1, 2
4 is the least natural number that has exactly 2 different proper factors.
Sum of factors of 4 = 7
Sum of proper factors of 4 = 3
4 is a deficient number.
4 is a square number. $4 = 2 \times 2 = 2^2$
4 is the sum of the first 2 odd numbers. 4 = 1 + 3
4 is a power of 2. $4 = 2^2$
Geometry: a quadrilateral has 4 sides and 4 vertices.
Geometry: a parallelogram has 4 sides and 4 vertices.
Geometry: a rectangle has 4 sides and 4 vertices.
Geometry: a rhombus has 4 sides and 4 vertices.
Geometry: a square has 4 sides and 4 vertices.
Geometry: a tetrahedron has 4 faces and 4 vertices.
A heffalump has 4 legs.
A beryllium (Be) atom has 4 protons.

5 (five)

5 is a natural number.
5 is the successor of 4.
5 is the predecessor of 6.
5 is an odd number.
5 is a prime number.
Factors of 5: 1, 5
Proper factor of 5: 1
Sum of factors of 5 = 6
Sum of proper factors of 5 = 1
5 is a deficient number.
5 is a Fibonacci number.
5 is the sum of the first 2 prime numbers. 5 = 2 + 3
5 is the sum of the first 2 square numbers. 5 = 1 + 4
Geometry: a pentagon has 5 sides and 5 vertices.
Geometry: a pyramid has 5 faces and 5 vertices.
A boron (B) atom has 5 protons.

6 (six)

6 is a natural number.
6 is the successor of 5.
6 is the predecessor of 7.
6 is an even number.
6 is a composite number.
Prime factorization: 6 = 2 x 3

Factors of 6: 1, 2, 3, 6
6 is the least number that has exactly 4 different factors.
Proper factors of 6: 1, 2, 3
6 is the least number that has exactly 3 different proper factors.
Sum of factors of 6 = 12
Sum of proper factors of 6 = 6
6 is a perfect number.
6 is the least perfect number.
6 is a factorial number. 6 = 3! = 1 x 2 x 3
6 is a triangular number. 6 = 1 + 2 + 3
Geometry: a hexagon has 6 sides and 6 vertices.
Geometry: a hexahedron has 6 faces.
A carbon (C) atom has 6 protons.

And so to 99. Now that we have discovered how interesting products of proper factors are, we itch to add that information to our mathemagical numbers eBooks.

End | TOC